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THE NON LINEAR CAUCHY PROBLEM OF OPERATOR DIFFERENTIAL EQUATIONS

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We consider the non linear Cauchy problem

$$X'(\lambda) = -sa(\lambda)\, X^{n+1}(\lambda), \qquad X(0) = I$$

over the field of Mikusiński operators; here $a(\lambda)$ is continuous numerical function and s is the operator of differentiation.

We construct a solution $X(\lambda)$ to the above Cauchy problem. This solution is continuous, possesses the continuous derivative for $\lambda \in [0, \Lambda]$, and belongs to the class $\mathcal{A}(f)$ of operator functions defined below (Definition 1).

Further we give several properties of the class $\mathcal{A}(f)$ which is of interest in its own right. Namely a class very similar to the $\mathcal{A}(f)$ occured earlier in connection with certain linear Cauchy problem of this kind [2].

Remark: Definitions and notations concerning Mikusiński operators can be found in [1]. Further, C is the set of all complex function of a real variable t, which are continuous for $t \geqslant 0$.

Definition 1. *$\mathcal{A}(f)$ is a subset of Mikusiński operational function of the form*

$$X(\lambda) = \begin{cases} \sum_{k=0}^{\infty} \dfrac{a_k\, l^{k+\gamma}}{f^{k+\alpha}(\lambda)} & 0 < \lambda \leqslant \Lambda \\ p & \lambda = 0, \end{cases}$$

here $\gamma \geqslant 0$ and $\alpha \geqslant 0$; l is the operator of integration, p is an operator, $f(\lambda)$ is a numerical function defined on $(0, \Lambda]$ and $f(\lambda) \neq 0$ for any $\lambda \in (0, \Lambda]$. a_k are complex numbers and the series $\sum_{k=0}^{\infty} a_k$ converges absolutely.

Lemma 1. *The operational series $\sum_{k=0}^{\infty} \dfrac{a_k\, l^{k+\gamma}}{f^{k+\alpha}(\lambda_0)}$ is convergent in the operational sense for any $\lambda_0 \in (0, \Lambda]$.*

Proof. First notice that

$$l\, \frac{a_k\, l^{k+\gamma}}{f^{k+\alpha}(\lambda_0)} = \left\{ \frac{a_k\, t^{k+\gamma}}{\Gamma(k+\gamma+1)\, f^{k+\alpha}(\lambda_0)} \right\} \in C.$$

For each $k=1, 2, 3, \ldots$ and $T>0$, one has $\Gamma(k+\gamma+1)>\left(\frac{2T}{m}\right)^{k+\gamma} e^{-\left(1+\frac{2T}{m}\right)}$ where $m=|f(\lambda_0)|$. Therefore, for $0<t\leqslant T$

$$\left|\frac{a_k t^{k+\gamma}}{\Gamma(k+\gamma+1) f^{k+\alpha}(\lambda_0)}\right| \leqslant \frac{|a_k| e^{1+\frac{2T}{m}} m^{\gamma-\alpha}}{2^{k+\gamma}}.$$

From the above inequality follows that the series $\sum_{k=0}^{\infty} \frac{a_k t^{k+\gamma}}{\Gamma(k+\gamma+1) f^{k+\alpha}(\lambda_0)}$ converges absolutely and uniformly in every finite interval $0\leqslant t\leqslant T$, which proves lemma 1.

Definition 2. *Let $X(\lambda)$ and $Y(\lambda)$ be two elements of $\mathcal{A}(f)$:*

$$X(\lambda)=\begin{cases} \sum_{k=0}^{\infty} \frac{a_k l^{k+\gamma}}{f^{k+\alpha}(\lambda)} & 0<\lambda\leqslant\Lambda \\ p & \lambda=0 \end{cases}$$

$$Y(\lambda)=\begin{cases} \sum_{k=0}^{\infty} \frac{b_k l^{k+\delta}}{f^{k+\beta}(\lambda)} & 0<\lambda\leqslant\Lambda \\ r & \lambda=0 \end{cases}$$

The operational function $Z(\lambda)$

$$Z(\lambda)=\begin{cases} \sum_{k=0}^{\infty} \frac{c_k l^{k+\gamma+\delta}}{f^{k+\alpha+\beta}(\lambda)} & 0<\lambda\leqslant\Lambda \\ pr & \lambda=0, \end{cases}$$

where $c_k=\sum_{i=0}^{k} a_i b_{k-i}$, is called the product ($\bigcirc$) of $X(\lambda)$ and $Y(\lambda)$, $Z(\lambda)=X(\lambda)\circ Y(\lambda)$.

Lemma 2. *For any $\lambda_0\in[0, \Lambda]$* holds:

1° $Z(\lambda_0)=X(\lambda_0)\, Y(\lambda_0)$

2° $Z(\lambda)\in\mathcal{A}(f)$

3° $X(\lambda)\circ Y(\lambda)=Y(\lambda)\circ X(\lambda)$

4° $X(\lambda)\circ[Y(\lambda)\circ T(\lambda)]=[X(\lambda)\circ Y(\lambda)]\circ T(\lambda)$.

Proof. 1° For $\lambda_0\neq 0$ we put

$$X(\lambda_0)=\sum_{k=0}^{\infty}\omega_k; \quad X_n(\lambda_0)=\sum_{k=0}^{n}\omega_k; \quad \omega_k=\frac{a_k l^{k+\gamma}}{f^{k+\alpha}(\lambda_0)}$$

$$Y(\lambda_0)=\sum_{k=0}^{\infty}w_k; \quad Y_n(\lambda_0)=\sum_{k=0}^{n}w_k; \quad w_k=\frac{b_k l^{k+\delta}}{f^{k+\beta}(\lambda_0)}$$

$$F = lX(\lambda_0) = \left\{ \sum_{k=0}^{\infty} \psi_k(t) \right\}, \qquad \{F_n(t)\} = \left\{ \sum_{k=0}^{n} \psi_k(t) \right\}$$

$$G = lY(\lambda_0) = \left\{ \sum_{k=0}^{\infty} \varphi_k(t) \right\}, \qquad \{G_n(t)\} = \left\{ \sum_{k=0}^{n} \varphi_k(t) \right\}$$

$$U(t) = \left\{ \sum_{k=0}^{\infty} |\psi_k(t)| \right\} \qquad V(t) = \left\{ \sum_{k=0}^{\infty} |\varphi_k(t)| \right\}$$

$$M = \max_{0 \leqslant t \leqslant T} U(t) \qquad L = \max_{0 \leqslant t \leqslant T} V(t)$$

(all series are convergent by Lemma 1).

Now we have

$$Z(\lambda_0) = \sum_{k=0}^{\infty} \frac{c_k \, l^{k+\gamma+\delta}}{f^{k+\alpha+\beta}(\lambda_0)} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \omega_i w_{k-i}$$

and

$$Z_n(\lambda_0) = \sum_{k=0}^{n} \sum_{i=0}^{k} \omega_i w_{k-i} = X_n(\lambda_0) \, Y(\lambda_0) + \sum_{k=0}^{n} \omega_k [y_{n-k}(\lambda_0) - y(\lambda_0)].$$

To complete the proof we have to show that for $n \to \infty$

$$l^2 \sum_{k=0}^{n} \omega_k [Y_{n-k}(\lambda_0) - Y(_0\lambda)] = \sum_{k=0}^{n} \{\psi_k(t)\} \{G_{n-k}(t) - G(t)\} \to 0$$

uniformly in every finite interval $0 \leqslant t \leqslant T$.

From Lemma 1. there follows that for any $\varepsilon > 0$, $0 \leqslant t \leqslant T$, $p = 1, 2, \ldots$ there exist N_1 and N_2 such that

$$|G_n(t) - G(t)| < \frac{\varepsilon}{2MT} \quad \text{for} \quad n > N_1$$

and

$$\sum_{i=n+1}^{n+p} |\psi_i(t)| < \frac{\varepsilon}{4KT} \quad \text{for} \quad n > N_2.$$

Let $N = \max(N_1, N_2)$; now for $n > 2N$ and $0 \leqslant t \leqslant T$ we have

$$\left| \sum_{k=0}^{n} \int_0^t \psi_k(t-\tau) [G_{n-k}(\tau) - G(\tau)] \, d\tau \right| \leqslant$$

$$\sum_{k=0}^{n-N-1} \int_0^t |\psi_k(t-\tau)| \, |G_{n-k}(\tau) - G(\tau)| \, d\tau +$$

$$\sum_{k=n-N}^{n} \int_0^t |\psi_k(t-\tau)| \, |G_{n-k}(\tau) - G(\tau)| \, d\tau \leqslant \varepsilon$$

Therefore $Z(\lambda_0) = X(\lambda_0) \, Y(\lambda_0)$ for $\lambda_0 \in [0, \Lambda]$.

Properties 2°, 3°, 4° are obvious.

L e m m a 3. *Suppose that* $f(\lambda)$ *is a numerical function defined on* $(0, \Lambda]$, *and* $f(\lambda)\neq 0$. *Further, put for* $0<\gamma<1$

$$X(\lambda)=\begin{cases}\gamma^{\gamma}\sum_{k=0}^{\infty}\binom{-\gamma}{k}\frac{l^{k+\gamma}\gamma^{k}}{f^{k+\gamma}(\lambda)} & 0<\lambda\leqslant\Lambda\\ I & \lambda=0\end{cases}$$

then $X(\lambda)\in\mathcal{A}(f)$ *and for each* $n\in N$ *holds*

$$X^{n}(\lambda)=\begin{cases}\gamma^{n\gamma}\sum_{k=0}^{\infty}\binom{-n\gamma}{k}\frac{l^{k+n\gamma}\gamma^{n}}{f^{k+n\gamma}(\lambda)} & 0<\lambda\leqslant\Lambda\\ I & \lambda=0.\end{cases}$$

P r o o f. The series $\sum_{k=0}^{\infty}\gamma^{k}\frac{(-1)^{k}\Gamma(\gamma+k)}{\Gamma(\gamma)k!}$ is absolutely convergent and so $X(\lambda)\in\mathcal{A}(f)$. The identity follows from definition 2. and from

$$\binom{\alpha+\beta}{k}=\sum_{i=0}^{k}\binom{\alpha}{i}\binom{\beta}{k-i}.$$

L e m m a 4. *Let* $f(\lambda)$ *be a continuous numerical function in* $[0, \Lambda]$, $f(\lambda)\neq 0$ *for* $\lambda\in(0, \Lambda]$, $f(0)=0$. *If there exists certain interval* $(0, \eta]$, $\eta\leqslant\Lambda$ *in which* $f(\lambda)>0$, *then the operational function*

$$U(\lambda)=\begin{cases}\sum_{k=0}^{\infty}\frac{\binom{-\gamma}{k}\gamma^{k}l^{k+\gamma}}{f^{k+\gamma}(\lambda)} & 0<\lambda\leqslant\Lambda, \quad 0<\gamma<1,\\ I & \lambda=0\end{cases}$$

is continuous in the closed interval $[0, \Lambda]$ *and* $U(\lambda)\in\mathcal{A}(f)$.

P r o o f. For $0<\gamma<1$ the series $\sum_{k=0}^{\infty}\binom{-\gamma}{k}\gamma^{k}$ is absolutely convergent so that $U(\lambda)\in\mathcal{A}(f)$. The continuity of $U(\lambda)$ in $[0, \Lambda]$ follows from the continuity of the numerical function $F(\lambda, t)$ of two variables λ, t such that $0\leqslant\lambda\leqslant\Lambda$, $t\geqslant 0$, where $F(\lambda, t)$ is defined by the parametric function

$$\{F(\lambda, t)\}=l^{1+\varepsilon}U(\lambda), \quad \varepsilon>0$$

$$F(\lambda, t)=\begin{cases}\frac{\gamma^{\gamma}}{\Gamma(1+\varepsilon)f^{\gamma}(\lambda)\Gamma(\gamma)}\int_{0}^{t}(t-\tau)^{\varepsilon}\tau^{\gamma-1}e^{-\frac{\gamma\tau}{f(\lambda)}}d\tau & 0<\lambda\leqslant\Lambda,\ t\geqslant 0\\ \frac{t^{\varepsilon}}{\Gamma(1+\varepsilon)} & \lambda=0,\ t\geqslant 0\end{cases}$$

$$F(\lambda, t)=\begin{cases}\frac{\gamma^{\gamma}t^{\varepsilon+\gamma}}{\Gamma(1+\varepsilon)f^{\gamma}(\lambda)\Gamma(\gamma)}\int_{0}^{1}(1-\tau)^{\varepsilon}\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}d\tau & 0<\lambda\leqslant\Lambda,\ t\geqslant 0\\ \frac{t^{\varepsilon}}{\Gamma(1+\varepsilon)} & \lambda=0,\ t\geqslant 0\end{cases}$$

The function of the real variables $(1-\tau)^{\varepsilon}\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}$ is continuous for $0<\tau\leqslant 1$, $t\geqslant 0$, $0<\lambda\leqslant\Lambda$.

Further integral $\int_0^1 (1-\tau)\,\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}d\tau$ is uniformly convergent in every closed domain $0\leqslant t\leqslant T$, $0<\delta\leqslant\lambda\leqslant\Lambda$. Therefore $F(\lambda, t)$ is continuous in the domain $t\geqslant 0$, $0<\lambda\leqslant\Lambda$.

When $\lambda\to 0$ and $t\to t_0\neq 0$ $\int_0^1 (1-\tau)^{\varepsilon}\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}d\tau\sim\Gamma(\gamma)\left(\frac{f(\lambda)}{\gamma t}\right)^{\gamma}$ thus we have

$$\lim_{\substack{\lambda\to 0\\ t\to t_0\neq 0}} F(\lambda, t)=\frac{t^{\varepsilon}_0}{\Gamma(1+\varepsilon)}.$$

To complete the proof we have to see that $\lim\limits_{\substack{\lambda\to 0\\ t\to 0}} F(\lambda, t)=0$ in the domain $0<\lambda\leqslant\Lambda$, $t\geqslant 0$. For each $\xi>0$ there exists a domain

$$0\leqslant t<\min\left(1, \frac{1}{2}\sqrt[\varepsilon]{\xi\,\Gamma(1+\varepsilon)}\right),\ 0<\lambda\leqslant\lambda_1<\eta \text{ in which}$$

$$|F(\lambda, t)|\leqslant\gamma^{\gamma}\frac{1}{\Gamma(1+\varepsilon)\Gamma(\gamma)}2^{\varepsilon}t^{\varepsilon}\frac{1}{f^{\gamma}(\lambda)}\int_0^1\tau^{\gamma-1}e^{-\frac{\gamma\tau}{f(\lambda)}}d\tau\leqslant\xi,$$

which proves the Lemma 4.

L e m m a 5. *Let* $f(\lambda)$ *be a continuous numerical function in* $[0, \Lambda]$, $f(\lambda)\neq 0$ *for* $\lambda\in(0, \Lambda]$, $f(0)=0$. *If there exists certain interval* $(0, \eta]$, $\eta<\Lambda$, *in which* $f(\lambda)\neq 0$, *then the operational function*

$$Y(\lambda)=\begin{cases}\sum\limits_{k=0}^{\infty}\binom{-\gamma}{k}\dfrac{\gamma^k\, l^{k+\gamma}(k+\gamma)}{f^{k+\gamma+1}(\lambda)} & 0<\lambda\leqslant\Lambda\\ s & \lambda=0\end{cases}$$

$0<\gamma<1$, *is continuous in the closed interval* $[0, \Lambda]$ *and* $Y(\lambda)\in\mathcal{A}(f)$.

P r o o f. For $0<\gamma<1$ the series $\sum\limits_{k=0}^{\infty}\binom{-\gamma}{k}\gamma^k(k+\gamma)$ is absolutely convergent and so $Y(\lambda)\in\mathcal{A}(f)$.

The continuity of $Y(\lambda)$ in $[0, \Lambda[$ follows from the continuity of the numerical function $G(\lambda, t)$ of two variables in the domain $0\leqslant\lambda\leqslant\Lambda$, $t\geqslant 0$; where $G(\lambda, t)$ is defined by the parametric function $\{G(\lambda, t)\}=l^3\,Y(\lambda)$

$$G(\lambda, t)=\begin{cases}\gamma^{\gamma}\dfrac{1}{\Gamma(3)\Gamma(\gamma)}\displaystyle\int_0^t (t-\tau)^2\tau^{\gamma-1}e^{-\frac{\gamma\tau}{f(\lambda)}}\left[\frac{\gamma}{f^{\gamma+1}(\lambda)}-\frac{\gamma\tau}{f^{2+\gamma}(\lambda)}\right]d\tau \\ \qquad 0<\lambda\leqslant\Lambda, \qquad t\geqslant 0 \\ \dfrac{t}{\Gamma(2)} \qquad \lambda=0 \qquad t\geqslant 0\end{cases}$$

$$G(\lambda, t)=\begin{cases}\dfrac{\gamma^{\gamma}t^{2+\gamma}}{\Gamma(3)\Gamma(\gamma)}\displaystyle\int_0^1 (1-\tau)^2\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}\left[\frac{\gamma}{f^{\gamma+1}(\lambda)}-\frac{\gamma t\tau}{f^{2+\gamma}(\lambda)}\right]d\tau, & 0<\lambda\leqslant\Lambda \\ & t\geqslant 0 \\ \dfrac{t}{\Gamma(2)} & \lambda=0 \\ & t\geqslant 0.\end{cases}$$

For $0<\tau\leqslant 1$, $0<\lambda\leqslant\Lambda$, $t\geqslant 0$ the function $(1-\tau)^2\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}\left[\frac{\gamma}{f^{\gamma+1}(\lambda)}-\frac{\gamma t\tau}{f^{\tau+\gamma}(\lambda)}\right]$ of the real variables is continuous.

Further integral $\int_0^1(1-\tau)^2\tau^{\gamma-1}e^{-\frac{\gamma t\tau}{f(\lambda)}}\left[\frac{\gamma}{f^{\gamma+1}(\lambda)}-\frac{\gamma t\tau}{f^{2+\gamma}(\lambda)}\right]d\tau$ is uniformly convergent in every closed domain $0\leqslant t\leqslant T$, $0<\delta\leqslant\lambda\leqslant\Lambda$. Therefore $G(\lambda, t)$ is continuous in the domain $0<\lambda\leqslant\Lambda$, $t\geqslant 0$.

$$\lim_{\substack{\lambda\to 0\\ t\to t_0\neq 0}} G(\lambda, t)=\lim_{\substack{\lambda\to 0\\ t\to t_0\neq 0}}\gamma^{\gamma}\frac{t^{2+\gamma}}{\Gamma(3)\Gamma(\gamma)}\left[\frac{\gamma}{f^{\gamma+1}(\lambda)}\Gamma(\gamma)\left(\frac{f(\lambda)}{\gamma t}\right)^{\gamma}-\right.$$

$$-\frac{2\gamma}{f^{\gamma+1}(\lambda)}\Gamma(1+\gamma)\left(\frac{f(\lambda)}{\gamma t}\right)^{1+\gamma}-\frac{\gamma t}{f^{2+\gamma}(\lambda)}\Gamma(1+\gamma)\left(\frac{f(\lambda)}{\gamma t}\right)^{1+\gamma}+$$

$$\left.+\frac{2\gamma t}{f^{2+\gamma}(\lambda)}\Gamma(2+\gamma)\left(\frac{f(\lambda)}{\gamma t}\right)^{2+\gamma}\right]=\frac{t_0}{\Gamma(2)}.$$

To complete the proof we have to see that $\lim\limits_{\substack{\lambda\to 0\\ t\to 0}} G(\lambda, t)=0$ in the domain $0<\lambda\leqslant\Lambda$, $t\geqslant 0$. For each $\xi>0$ there exists a domain $0<\lambda\leqslant\lambda_1<\eta$, $0\leqslant t<$ $<\min\left(1, \frac{\xi}{2}\right)$ such that $f(\lambda)<\frac{\xi}{2}$, in which

$$|G(\lambda, t)|\leqslant\frac{\gamma^{\gamma+1}}{f^{2+\gamma}(\lambda)\Gamma(3)\Gamma(\gamma)}\left(\frac{2tf(\lambda)}{\gamma}\int_0^1\tau^{\gamma}e^{-\frac{\gamma\tau}{f(\lambda)}}d\tau+\right.$$

$$\left.+\frac{2f(\lambda)}{\gamma}\int_0^1\tau^{1+\gamma}e^{-\frac{\gamma\tau}{f(\lambda)}}d\tau\right)\leqslant\xi,$$

which proves the Lemma 5.

Lemma 6. *If $f(\lambda)$ is differentiable in $(0, \Lambda]$, the operational functions of the class $\mathcal{A}(f)$ are also differentiable in $(0, \Lambda]$ and one can obtain its derivative by differentiating the defining operational series term by term.*

Proof. Let $\omega(\lambda) \in \mathcal{A}(f)$. For $0 < \lambda \leqslant \Lambda$, $\omega(\lambda) = \sum_{k=0}^{\infty} \frac{a_k l^{k+\gamma}}{f^{k+\alpha}(\lambda)}$. Then the numerical function $0(\lambda, t)$ defined by the parametrical function $\{0(\lambda, t)\} = l\,\omega(\lambda)$ is of the form $0(\lambda, t) = \sum_{k=0}^{\infty} \frac{a_k t^{k+\gamma}}{\Gamma(k+\gamma+1) f^{k+\alpha}(\lambda)}$ $0 < \lambda \leqslant \Lambda$.

The series $\sum_{k=0}^{\infty} \frac{a_k t^{k+\gamma}(k+\alpha)}{\Gamma(k+\gamma+1) f^{k+\alpha}(\lambda)}$ is uniformly convergent in each closed domain $0 < \lambda_1 \leqslant \lambda \leqslant \Lambda$, $0 \leqslant t \leqslant T$. Hence, for $0 < \lambda \leqslant \Lambda$

$$\frac{\partial 0(\lambda, t)}{\partial \lambda} = -\sum_{k=0}^{\infty} \frac{a_k t^{k+\gamma}(k+\alpha)}{\Gamma(k+\gamma+1) f^{k+\alpha+1}(\lambda)} f'(\lambda).$$

Therefore for $0 < \lambda \leqslant \Lambda$

$$\omega'(\lambda) = -\sum_{k=0}^{\infty} \frac{a_k (k+\alpha) l^{k+\gamma}}{f^{k+\alpha+1}(\lambda)} f'(\lambda).$$

Lemma 7. *Let $f(\lambda)$ be a continuous numerical function in $[0, \Lambda]$ $f(\lambda) \neq 0$ for $\lambda \in (0, \Lambda]$, $f(0) = 0$. Further, let $f(\lambda)$ possess a bounded derivation $f'(\lambda)$ in $[0, \Lambda]$. If there exists certain interval $(0, \eta]$, $\eta \leqslant \Lambda$ in which $f(\lambda) > 0$, then the operational function*

$$U(\lambda) = \begin{cases} \gamma^{\gamma} \sum_{k=0}^{\infty} \frac{\binom{-\gamma}{k} \gamma^k l^{k+\gamma}}{f^{k+\gamma}(\lambda)} & 0 > \lambda \leqslant \Lambda \\ I & \lambda = 0 \end{cases}$$

$0 < \gamma < 1$; *is differentiable in $[0, \Lambda]$, $U(\lambda) \in \mathcal{A}(f)$ and*

$$U'(\lambda) = \begin{cases} -\gamma^{\gamma} \sum_{k=0}^{\infty} \binom{-\gamma}{k} \gamma^k \frac{l^{k+\gamma}(k+\gamma) f'(\lambda)}{f^{k+\gamma+1}(\lambda)} & 0 < \lambda \leqslant \Lambda \\ -s f'(0) & \lambda = 0 \end{cases}$$

Proof. The relation $U(\lambda) \in \mathcal{A}(f)$ is proved in Lemma 4, and from Lemma 6 follows

$$U'(\lambda) = -\gamma^{\gamma} \sum_{k=0}^{\infty} \frac{\binom{-\gamma}{k} \gamma^k l^{k+\gamma}(k+\gamma) f'(\lambda)}{f^{k+\gamma+1}(\lambda)} \qquad 0 < \lambda \leqslant \Lambda.$$

We have therefore only to show that $U'(0) = -s f'(0)$. Consider the numerical function $H(\lambda, t)$ of two variables λ, t, $0 \leqslant \lambda \leqslant \Lambda$, $t \geqslant 0$ defined by the parametrical function $\{H(\lambda, t)\} = l^{1+\varepsilon} U(\lambda)$, $\varepsilon > 0$, i.e.

$$H(\lambda, t) = \begin{cases} \int_0^t \frac{(t-\tau)^{\varepsilon}}{\Gamma(1+\varepsilon)} \frac{\gamma^{\gamma} \tau^{\gamma-1} e^{-\frac{\tau\gamma}{f(\lambda)}}}{f^{\gamma}(\lambda) \Gamma(\gamma)} d\tau & 0 < \lambda \leqslant \Lambda \quad t \geqslant 0 \\ \frac{t^{\varepsilon}}{\Gamma(1+\varepsilon)} & \lambda = 0 \quad t \geqslant 0 \end{cases}$$

For $\left.\frac{\partial H(\lambda, t)}{\partial \lambda}\right|_{\lambda=0,\ t\neq 0}$ one obtains by definition that

$$\frac{\partial H(\lambda, t)}{\partial \lambda}=\lim_{\lambda\to 0}\frac{1}{\lambda}\left(\int_0^t \frac{(t-\tau)^{\varepsilon}}{\Gamma(1+\varepsilon)}\gamma^{\gamma}\tau^{\gamma-1}\frac{e^{-\frac{\tau\gamma}{f(\lambda)}}}{f^{\gamma}(\lambda)\Gamma(\gamma)}d\tau-\frac{t^{\varepsilon}}{\Gamma(1+\varepsilon)}\right)=$$

$$=\lim_{\lambda\leftarrow 0}\frac{1}{\lambda}\left[\gamma^{\gamma}\frac{t^{\varepsilon+\gamma}}{\Gamma(1+\varepsilon)f^{\gamma}(\lambda)\Gamma(\gamma)}\left(\Gamma(\gamma)\frac{f^{\gamma}(\lambda)}{\gamma^{\gamma}t^{\gamma}}-\Gamma(1+\gamma)\,\varepsilon\,\frac{f^{\gamma+1}(\lambda)}{\gamma^{\gamma+1}t^{\gamma+1}}\right)-\frac{t^{\varepsilon}}{\Gamma(1+\varepsilon)}\right]=$$

$$=-\frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)}f'(0),\ \text{and for}\ \left.\frac{\partial H(\lambda, t)}{\partial \lambda}\right|_{\lambda=0\ t=0}=0.$$

Then $U'(0)=sf'(0)$.

Theorem 1. *Let $a(\lambda)$ be a continuous numerical function in $[0, \Lambda]$ with the properties*

1) $\int_0^{\lambda} a(t)\,dt\neq 0$ *for* $\lambda\neq 0$

2) *There exists an* $\eta\in(0, \Lambda]$ *such that* $\int_0^{\lambda} a(t)\,dt>0$ *for* $\lambda<\eta$, *then the initial value problem*

$$(*)\quad \begin{aligned} &X'(\lambda)=-sa(\lambda)X^{n+1}(\lambda) \qquad (n=2,\ 3,\ 4,\dots)\\ &X(0)=I\end{aligned}$$

has in $[0, \Lambda]$ *the solution of the form:*

$$X(\lambda)=\begin{cases}\dfrac{1}{\sqrt[n]{n}}\displaystyle\sum_{k=0}^{\infty}\frac{\binom{-\frac{1}{n}}{k}I^{k+1/n}}{n^{k}\left[\int_0^{\lambda}a(t)\,dt\right]^{k+1/n}} & 0<\lambda\leqslant\Lambda\\ I & \lambda=0\end{cases}$$

$X(\lambda)$ *and* $X'(\lambda)$ *are continuous in* $[0, \Lambda]$ *and* $X(\lambda)\in\mathcal{A}\left(\int_0^{\lambda}a(t)\,dt\right)$.

Proof. If we choose in the lemmas $\gamma=\frac{1}{n}$ and $f(\lambda)=\int_0^{\lambda}a(t)\,dt$ then the continuity of $X(\lambda)$ and $X(\lambda)\in\mathcal{A}(\int_0^{\lambda}a(t)\,dt)$ follow from Lemma 4. The form of

$$X'(\lambda)=\begin{cases}-\dfrac{1}{\sqrt[n]{n}}\displaystyle\sum_{k=0}^{\infty}\frac{\binom{-\frac{1}{n}}{k}I^{k+1/n}(k+1/n)\,a(\lambda)}{n^{k}\left[\int_0^{\lambda}a(t)\,dt\right]^{k+1+1/n}} & 0<\lambda\leqslant\Lambda\\ -sa(0) & \lambda=0\end{cases}$$

and its continuity follows from Lemmas 5, 6, 7. Using Lemma 3 and the above expression for $X'(\lambda)$, the fact that $X(\lambda)$ is a solution of $(*)$ follows by inspection, and $X(0)=I$ is obvious.

Remark: For $n=1$ the theorem is valid too, but the solution is of the form

$$(\lambda)\,X=\begin{cases}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k\, l^{k+1}}{\left[\int_0^{\lambda} a(t)\,dt\right]^{k+1}} & 0<\lambda<\Lambda\\ I & \lambda=0\end{cases}$$

and so $X(\lambda)\notin\mathcal{A}\left(\int_0^{\lambda} a(t)\,dt\right)$.

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