PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 10 (24), 1970, pp. 125—131

THE CONTINUITY OF ONE CLASS OF OPERATIONAL FUNCTIONS

D. Nikolić-Despotović

(Communicated December 26, 1969)

The purpose of the woık is to research the continuity in the point $\lambda=\lambda_0$ of operational function $R(\lambda)$ which is defined by the relation:

$$[R(\lambda)]^n=\frac{I}{\alpha(\lambda)s+\beta(\lambda)} \qquad n\in N.$$

In that sense we shall write:

$$R(\lambda)=\frac{I}{(\alpha(\lambda)s+\beta(\lambda))^{\frac{1}{n}}}$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ are numerical, continuous functions on the interval $[\lambda_1, \lambda_2]$ and $\alpha(\lambda_0)=0$; s is the differential operator and I is the unit element. Multiplication $R^n(\lambda)=\underbrace{R(\lambda)R(\lambda)\cdots R(\lambda)}_{n}$ is in sense of multiplication of operators which are defined by Mikusiński's Operational Calculus [1].

This work partly extends the results obtained in [2]; they can be deduced from our theorem 1 for $n=1$.

In the proof of theorem 1, I shall use Mikusiński's theorem on bounded moments [3] i.e.

Theorem A; *If $\beta_1, \beta_2, \ldots$ is a sequence of positive numbers such that: $\sum_{n=1}^{\infty}\frac{1}{\beta_n}=\infty$ and $\beta_{n+1}-\beta_n>\varepsilon>0$, for $n=1,2,\ldots$ and $g(t)$ is a function, integrable in $[0, T]$, such that:*

$$\left|\int_0^T e^{\beta_n t} g(t)\,dt\right|<M$$

then $g(t)=0$ almost everywhere in $[0, T]$.

Theorem 1. *Suppose that:*

1. *$\alpha(\lambda)$ and $\beta(\lambda)$ are numerical, real, continuous functions on interval $[\lambda_1, \lambda_2]$.*
2. *$\lambda_0\in[\lambda_1, \lambda_2]$ and λ_0 is an isolated zero of the function $\alpha(\lambda)$, namely $\alpha(\lambda_0)=0$.*
3. $\beta(\lambda_0)\neq 0$.

Necessary and sufficient condition for the operational function

(1) $$R(\lambda)=\frac{I}{(\alpha(\lambda)s+\beta(\lambda))^{\frac{1}{n}}} \qquad n\in N$$

to be continuous in $\lambda=\lambda_0$, *in the sense of continuity in Operational Calculus, is the existence of a neighbourhod* V_0 *of the point* λ_0 *in which* $\frac{\beta(\lambda)}{\alpha(\lambda)}>0$ *while* $\lambda\in V_0/\{\lambda_0\}$.

Proof: The condition is sufficient

For $p\in N$ is

$$R(\lambda)=\begin{cases}\dfrac{1}{(\alpha(\lambda))^{\frac{1}{n}}}\dfrac{t^{\frac{1}{n}-1}}{\Gamma\left(\frac{1}{u}\right)} & \lambda\neq\lambda_0\\[2ex] \dfrac{1}{(\beta(\lambda))^{\frac{1}{n}}}I & \lambda=\lambda_0\end{cases}$$

$$=s^{p+1}\begin{cases}\dfrac{1}{(\alpha(\lambda))^{\frac{1}{n}}\Gamma\left(\frac{1}{n}\right)p!}\displaystyle\int_0^t e^{-\frac{\beta(\lambda)}{\alpha(\lambda)}u}\,u^{\frac{1}{n}}(t-u)^p\,du & \lambda\neq\lambda_0\\[2ex] \dfrac{t^p}{p!(\beta(\lambda))^{\frac{1}{n}}} & \lambda=\lambda_0\end{cases}$$

Supposing that there is a neighbourhood V_0 of the point λ_0 in which $\frac{\beta(\lambda)}{\alpha(\lambda)}=\gamma(\lambda)>0$ while $\lambda\in V_0\setminus\{\lambda_0\}$, then by substitution $\gamma(\lambda)u=x^n$ we obtain the numerical function:

$$P(\lambda,t)=\frac{1}{(\alpha(\lambda))^{\frac{1}{n}}p!\,\Gamma\left(\frac{1}{n}\right)}\int_0^t e^{-\gamma(\lambda)u}\,u^{\frac{1}{n}-1}(t-u)^p\,du=$$

$$=\frac{n}{(\beta(\lambda))^{\frac{1}{n}}p!\,\Gamma\left(1+\frac{1}{n}\right)}\int_0^{\sqrt[n]{\gamma(\lambda)t}}\left(t-\frac{x^n}{\gamma(\lambda)}\right)^p e^{-x^n}\,dx$$

$$P(\lambda,t)=\frac{1}{(\beta(\lambda))^{\frac{1}{n}}p!\,\Gamma\left(1+\frac{1}{n}\right)}\int_0^{\sqrt[n]{\gamma(\lambda)t}} t^p e^{-x^n}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\frac{x^{nk}}{\gamma_{(\lambda)}^k t^k}\,dx.$$

If we denote with:

$$I_n = \int_0^{\sqrt[n]{\gamma(\lambda)t}} e^{-x^n}\,dx, \quad \text{and with} \quad I_n^k = \int_0^{\sqrt[n]{\gamma(\lambda)t}} x^{nk} e^{-x^n}\,dx$$

then integrating by parts, we obtain

$$I_n^k = \frac{I_n}{n^k}\prod_{\nu=1}^{k-1}(n\nu+1) - \frac{1}{n}\, e^{-\gamma(\lambda)t}(\gamma(\lambda)t)^{1/n}\left\{(\gamma(\lambda)t)^{k-1} + \right.$$

$$\left. + \sum_{\nu=1}^{k-1}\left[(\gamma(\lambda)t)^{\nu-1}\prod_{\nu}^{k-1}\left(\nu+\frac{1}{n}\right)\right]\right\}$$

for $k = 1, 2, \ldots, p$.

Therefore

$$P(\lambda, t) = \frac{1}{(\beta(\lambda))^{\frac{1}{n}}\, p!\, \Gamma\left(1+\frac{1}{n}\right)}\left[(t^p - Q(\lambda, t))\, I_n + T(\lambda, t)\right] \tag{2}$$

where it is written down as:

$$Q(\lambda, t) = \frac{1}{n\gamma(\lambda)}\left[p t^{p-1} - \frac{n+1}{n\gamma(\lambda)}\binom{p}{2} t^{p-2} + \frac{(2n+1)(n+1)}{(n\gamma(\lambda))^2}\binom{p}{3} t^{p-3} + \cdots \right.$$

$$\left. + \cdots + \left(\frac{-1}{n\gamma(\lambda)}\right)^{p-1}\prod_{\nu=1}^{p-1}(n\nu+1)\right] \tag{3}$$

$$T(\lambda, t) = \frac{t^{1/n} e^{-\gamma(\lambda)t}}{n\gamma^{1-\frac{1}{n}}}\left\{t^{p-1} + \frac{t^{p-2}}{n\gamma(\lambda)}\sum_{k=1}^{p-1}(-1)^k (nk+1)\binom{p}{k+1} + \right.$$

$$+ \frac{t^{p-3}}{(n\gamma(\lambda))^2}\sum_{k=2}^{p-1}(-1)^k (nk+1)(nk-n+1)\binom{p}{k+1} + \cdots$$

$$\left. + \cdots + \frac{(-1)^{p-1}}{(n\gamma(\lambda))^{p-2}}\prod_{\nu=1}^{p-1}(n\nu+1)\right\}.$$

The operational function (1) has the following form:

$$R(\lambda) = s^{p+1}\begin{cases} \dfrac{1}{\beta^{\frac{1}{n}}\,\Gamma(1+1/n)\, p!}\left[(t^p - Q(\lambda, t))\, I_n + T(\lambda, t)\right] & \lambda \neq \lambda_0 \\[2ex] \dfrac{t^p}{p!\,\beta^{\frac{1}{n}}(\lambda)} & \lambda = \lambda_0 \end{cases}$$

where $Q(\lambda, t)$ and $T(\lambda, t)$ are defined by the equalities (3).

When $\lambda \to \lambda_0$ independent from t, then:

$$Q(\lambda, t) \to Q(\lambda_0, t) \equiv 0$$

$$T(\lambda, t) \to T(\lambda_0, t) = 0$$

$$I_n \to \int_0^\infty e^{-x^n}\, dx = \frac{1}{n}\, \Gamma\left(\frac{1}{n}\right) = \Gamma\left(1 + \frac{1}{n}\right)$$

and the numerical function

$$\begin{cases} \dfrac{1}{p!\, \beta^{\frac{1}{n}}(\lambda)\, \Gamma\left(1 + \dfrac{1}{n}\right)} \left[(t^p - Q(\lambda, t))\, I_n + T(\lambda, t)\right] & \lambda \neq \lambda_0 \\ \dfrac{t^p}{p!\, \beta^{\frac{1}{n}}(\lambda)} & \lambda = \lambda_0 \end{cases}$$

is continuous for $\lambda = \lambda_0$, $0 \leqslant t < \infty$ and by the definition of continuity of the operational function, $R(\lambda)$ is continuous in $\lambda = \lambda_0$. Thus the condition is sufficient.

The condition is necessary. Supposing now that there is not a neighbourhood of the point λ_0 in which $\frac{\beta(\lambda)}{\alpha(\lambda)} > 0$; then there will exist a closed neighbourhood $\overline{V}(\lambda_0)$ of the point λ_0 in which $\beta(\lambda)$ does not change its sign and reaches the maximum and minimum. The characteristic of a continuous function $\delta(\lambda) = -\frac{\beta(\lambda)}{\alpha(\lambda)}$ is that: for each neighbourhood $V_n(\lambda_0)$ of the point λ_0 there is a point $\lambda_n \in V_n(\lambda_0)$ such that $\delta(\lambda_n) > 0$. Let the neighbourhood $V_n(\lambda_0)$ make a monotonous basis; the sequence $\{\lambda_n\}$ converges to λ_0 and beginning from one n_0 $V_n(\lambda_0) \subset \overline{V}(\lambda_0)$ for every $n \geqslant n_0$. According to the supposition that $\beta(\lambda)$ on $\overline{V}(\lambda_0)$ (has the minimum on $\overline{V}$ different from zero and maximum) and knowing that $\alpha(\lambda_0) = 0$, it follows that $\delta(\lambda_n) \to \infty$ when $n \to \infty$.

Because of continuity of $\delta(\lambda)$ there is a subset of $\overline{V}(\lambda_0)$ which is mapped onto a halfline $x > \delta(\lambda_{n_0})$. There is, also a subset in each $V_n(\lambda_0)$ which is mapped onto a halfline $x > \delta(\lambda_n)$. Let $\delta(\lambda_{n_0})$ be such that:

$$m \leqslant \delta(\lambda_0) < m + 1$$

Let us form a sequence λ_i' in the following way:

$$\delta(\lambda_i') = m + i; \quad k = \max n, \text{ for which } \lambda_i' \in V_n(\lambda_0),\ \lambda_i' \in V_k(\lambda_0).$$

As $\delta(\lambda)$ is a continuous function it will also assume all the values between $\delta(\lambda'_{i+1})$ and $\delta(\lambda_i')$, while $\lambda_i' < \lambda < \lambda'_{i+1}$ and the sequence $\delta_i = \delta(\lambda_i')$ satisfies the conditons of the Theorem A. If in this cas the operational function would be continuous too, there would exist $f \in C$, $f \neq 0$, such that $R(\lambda) f$ is a numerous function of two variables λ and t, continuous for $\lambda=\lambda_0$ and $t\geqslant 0$. Namely, for each fixed number $T\in R^+$ there would exist a fixed number M such that:

$$\left|\int_0^T e^{\delta_1 u}\, u^{1/n-1} f(T-u)\, du\right| < M.$$

Function $g(u)=u^{1/n-1} f(T-u)$ is integrable in the interval $[0, T]$, therefore the conditions of Theorem A are satisfied, and according to the same theorem, it would follow that $g(u)=0$, almos everywhere, while $u\in[0, T]$; namely $f(u)=0$, almost everywhere, while $u\in[0, T]$. It is contradictory to the supposition that $f\not\equiv 0$, therefore the supposition from which we started does not hold, namely $R(\lambda)$ is not continuous at the point $\lambda=\lambda_0$.

T h e o r e m 2. *Let* $\alpha(\lambda)$ *and* $\beta(\lambda)$ *have the first derivate as numerical functions in the interval* $[\lambda_1, \lambda_2]$ *and satisfy the conditions of the preceding theorem. Necessary and sufficient condition that operational function* (1) *has the first derivate continuous at the point* $\lambda=\lambda_0$, *that there exists a neighbourhood* V_0 *of the point* λ_0 *in which* $\dfrac{\beta(\lambda)}{\alpha(\lambda)}>0$ *while* $\lambda\in V_0\setminus\{\lambda_0\}$.

P r o o f. The condition is sufficient:
If there exists a neighbourhood $V_0(\lambda_0)$ such that $\dfrac{\beta(\lambda)}{\alpha(\lambda)}>0$ while $\lambda\in V_0\setminus\{\lambda_0\}$ the operational function (1) is, according to the preceding theorem, continuous at point $\lambda=\lambda_0$, and can be written as:

$$R(\lambda)=s^{p+1}\begin{Bmatrix} P(\lambda, t) & \lambda\neq\lambda_0 \\ \dfrac{t}{p!\,\beta^{\frac{1}{n}}(\lambda)} & \lambda=\lambda_0 \end{Bmatrix}; \qquad \begin{Bmatrix} P(\lambda, t) & \lambda\neq\lambda_0 \\ \dfrac{t^p}{p!\,\beta^{\frac{1}{n}}(\lambda)} & \lambda=\lambda_0 \end{Bmatrix} \text{ is a continuous}$$

function for $\lambda=\lambda_0$, $0<t<\infty$. Therefore:

$$R'(\lambda)=s^{p+1}\begin{Bmatrix} \dfrac{\partial P(\lambda, t)}{\partial\lambda} & \lambda\neq\lambda_0 \\ \dfrac{-t^p\beta'(\lambda)}{p!\,n\,\beta^{1+\frac{1}{n}}(\lambda)} & \lambda=\lambda_0 \end{Bmatrix}$$

It should be shown that $\begin{Bmatrix} \dfrac{\partial P(\lambda, t)}{\partial\lambda} & \lambda\neq\lambda_0 \\ \dfrac{-t^p\beta'(\lambda)}{np!\,\beta^{1+\frac{1}{n}}(\lambda)} & \lambda=\lambda_0 \end{Bmatrix}$ is continuous for

$\lambda=\lambda_0$, $0\leqslant t<\infty$.

$$\frac{\partial P}{\partial\lambda}=\frac{1}{p!\,\Gamma\left(1+\frac{1}{n}\right)}\left[\frac{-\beta'(\lambda)}{n\beta_{(\lambda)}^{1+\frac{1}{n}}}(t^p-Q(\lambda,t)\,I_n-\frac{\beta'(\lambda)}{n\beta_{(\lambda)}^{1+\frac{1}{n}}}T(\lambda,t)+\right.$$

$$\left.+\frac{1}{\beta_{(\lambda)}^{\frac{1}{n}}}\frac{\partial T(\lambda,t)}{\partial\lambda}-\frac{\partial Q(\lambda,t)}{\beta_{(\lambda)}^{\frac{1}{n}}\partial\lambda}I_n+\frac{\gamma'(\lambda)}{n\gamma_{(\lambda)}^{1-\frac{1}{n}}}\left(\frac{t}{\beta(\lambda)}\right)^{1/n}(t^p-Q(\lambda,t))\right]$$

$$\frac{\partial P}{\partial\lambda}=\frac{1}{p!\,\Gamma\left(1+\frac{1}{n}\right)}\left\{\frac{-t^p\beta'(\lambda)\,I_n}{n\beta_{(\lambda)}^{1+\frac{1}{n}}}-\frac{\beta'(\lambda)}{n\beta_{(\lambda)}^{1+\frac{1}{n}}}T(\lambda,t)+\right.$$

$$+\frac{\gamma'(\lambda)\,t^{\frac{1}{n}}e^{-\gamma(\lambda)t}}{n\gamma_{(\lambda)}^{2-\frac{1}{n}}}S(t,\lambda)+\frac{I_n}{n\beta_{(\lambda)}^{\frac{1}{n}}}\left[\left(\frac{\beta'(\lambda)}{n\beta(\lambda)}+\frac{\gamma'(\lambda)}{\gamma^2(\lambda)}\right)p\,t^{p-1}-\right.$$

$$-\frac{n+1}{n\gamma^2(\lambda)}\binom{p}{2}t^{p-2}\left(\frac{\beta'(\lambda)}{n\beta(\lambda)}+\frac{2\gamma'(\lambda)}{\gamma(\lambda)}\right)-$$

$$\left.\left.+\cdots+(-1)^p\frac{\left(\frac{\beta'(\lambda)}{n\beta(\lambda)}+\frac{p\gamma'(\lambda)}{\gamma(\lambda)}\right)}{n^{p-1}\gamma_{(\lambda)}^p}\prod_{\nu=1}^{p-1}(n\nu+1)\right]\right\}$$

where

$$S(\lambda,t)=\frac{t^{p-1}}{n}\left[1-n-p-\sum_{k=2}^{p}(-1)^{k+1}(nk-n+1)\binom{p}{k}\right]+$$

$$+\frac{t^{p-2}}{n\gamma(\lambda)}\left[(n+1)\,p\,(p-1)+\sum_{k=3}^{p}(-1)^{k+1}(2-k)(nk-n+1)\binom{p}{k}\right]+ \tag{4}$$

$$+\cdots+\frac{(-1)^p\left(\frac{2}{n}-p\right)\prod_{\nu=1}^{p-1}(n\nu+1)}{(n\gamma(\lambda))^{p-1}}$$

When $\lambda\to\lambda_0$ indenpendent of t, then for $p-1\geqslant 1$

$$\frac{\partial P}{\partial\lambda}\to\frac{-t^p\beta'(\lambda_0)\,\Gamma(1+1/n)}{np!\,\beta_{(\lambda_0)}^{1+\frac{1}{n}}\Gamma(1+1/n)}-\frac{p\,t^{p-1}\Gamma(1+1/n)\,\alpha'(\lambda_0)}{p!\,\beta^{\frac{1}{n}}(\lambda_0)\,\Gamma(1+1/n)}.$$

Namely

$$R'(\lambda)\to -s^{p+1}\,l^{p+1}\left(\frac{\beta'(\lambda_0)}{n\beta_{(\lambda_0)}^{\frac{1}{n}+1}}\right)-s\left(\frac{\alpha'(\lambda_0)}{\beta^{\frac{1}{n}}(\lambda_0)}\right)=-\frac{\beta'(\lambda_0)}{n\beta_{(\lambda_0)}^{\frac{1}{n}+1}}=R'(\lambda_0)$$

The condition is necessary.

If does not exist a neighbourhood V_0 of the point λ_0 in which $\frac{\beta(\lambda)}{\alpha(\lambda)} > 0$ while $\lambda \in V_0/\{\lambda_0\}$, then according to the preceeding theorem, the function (1) is not continuous at $\lambda = \lambda_0$, and cannot have the first derivate continuous at the point $\lambda = \lambda_0$. So the condition of necessity of the theorem is proved.

REFERENCES

[1] Mikusiński, J., *Operational Calculus,* Pergamon Press, New York, 195.

[2] D. Despotović, B. Stanković, *Continuité d'une fonction opératoire,* Publ. Inst. Math., Beograd, 7 (1967) 197-203.

[3] Mikusiński, J., Nardzewski R., *A theorem on bounded moments,* Studia Math., XIII (1953) 51-55.