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ON THE FUNCTION OF E. M. WRIGHT

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When one applies the Laplace transform, the following integral function:

$$\Phi(\beta, -\sigma; z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\,\Gamma(\beta - n\sigma)}, \quad 0<\sigma<1 \tag{1}$$

or a special case of it, appears. Mikusiński's operators point more and more to its increasing importance. A number of authors have used some functions which were really simply Wright's function or some particular instance of it.

The aim of this paper is to show that we have here only different forms of the same function. For this reason we shall use a notation independent of the notations used by different authors. But the main intention is both to give the known results and to prove the new results of this function. We shall only give proofs for the new results. For results known, an outline of the proof will be given or an indication where the proof can be found.

We hope that this paper will be useful to all working with this function.

1. *The function Φ given by an integral in the complex plane.*

The first author who sistematically studied this function was E. M. Wright [14, 15] He also gave its integral form.

We denote by C a contour in the u-plane starting from infinity in the part on the plane in which $-\frac{3}{2}\pi + \frac{1}{2}\varepsilon < \arg u < -\frac{1}{2}\pi - \frac{1}{2}\varepsilon$, passing round the origin to in a counter-clockwise direction, and going to infinity in the part of the plane in which $\frac{1}{2}\pi + \frac{1}{2}\varepsilon < \arg u < \frac{3}{2}\pi - \frac{1}{2}\varepsilon$ (Fig. 1).

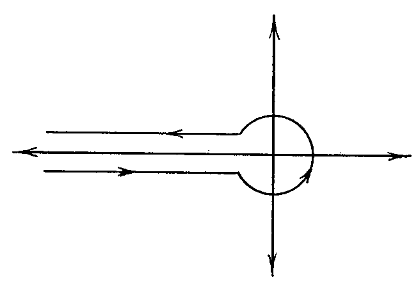

Fig. 1

Then:

$$\frac{1}{\Gamma(-\sigma n + \beta)} = \frac{1}{2\pi i} \int_C u^{\sigma n - \beta} e^u \, du$$

and from (1)

$$\Phi(\beta, \ -\sigma; \ z) = \frac{1}{2\pi i} \int_C u^{-\beta} \exp(u + zu^{\sigma})\, du. \tag{2}$$

The change of integration and summation order being readily justified. For the function z^{σ} it being understood that z^{σ} has its principal branch.

2. *The particular cases of* Φ

We shall cite some particular cases:

1. $z = -x^{-\sigma}$, x being a real and positive number

$$x^{\beta-1}\Phi(\beta, \ -\sigma; \ -x^{-\sigma}) = \frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} s^{-\beta} e^{xs-s^{\sigma}}\, ds, \quad x_0 > 0. \tag{3}$$

We should only start from the integral [8]:

$$\int_{C'} s^{-\beta} \exp(xs - s^{\sigma})\, ds = 0, \tag{4}$$

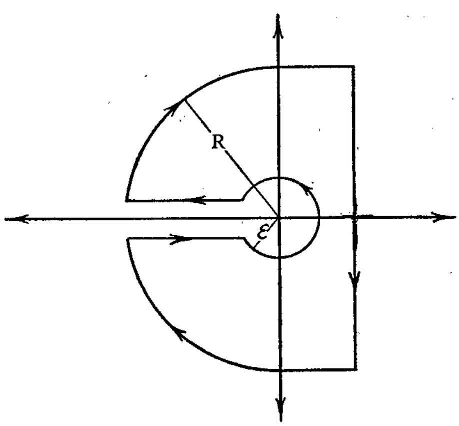

Fig. 2

where the contour C' is given in Fig. 2

Here we have:

$$\frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} s^{-\beta} \exp(xs - s^{\sigma})\, ds =$$

$$= \frac{1}{2\pi i} \int_C s^{-\beta} \exp(xs - s^{\sigma})\, ds,$$

$$= x^{\beta-1} \frac{1}{2\pi i} \int u^{-\beta} \exp(u - x^{-\sigma} u^{\sigma}) du$$

$$= x^{\beta-1}\Phi(\beta, \ -\sigma; \ -x^{-\sigma}), \ x_0 > 0.$$

2. $\beta < 1$.

We shall start from the same integral (4), and take the limits when $R \to \infty$, and $\varepsilon \to 0$:

$$\frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} s^{-\beta} \exp(ts - s^{\sigma})\, ds = \frac{1}{2\pi i} \int_0^{\infty} x^{-\beta} \exp(\beta\pi i - tx - x^{\sigma} e^{-\sigma\pi i})\, dx -$$

$$-\frac{1}{2\pi i} \int_0^{\infty} x^{-\beta} \exp(-\beta\pi i - tx - x^{\sigma} e^{\sigma\pi i})\, dx$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-tx} x^{-\beta} e^{-x^{\sigma}\cos\sigma\pi} \sin(x^{\sigma}\sin\sigma\pi + \beta\pi)\, dx.$$

Compared with (3) we have;

$$t^{\beta-1}\Phi(\beta, -\sigma; -t^{-\sigma}) = \frac{1}{\pi}\int_0^\infty e^{-tx} x^{-\beta} e^{-x^\sigma \cos \sigma\pi} \sin(x^\sigma \sin \sigma\pi + \beta\pi)\, dx. \tag{5}$$

This limit-processes show that in (3) we can choose $x_0 = 0$ when $\beta < 1$. So we have:

$$x^{\beta-1}\Phi(\beta, -\sigma; -x^{-\sigma}) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} s^{-\beta} e^{xs - s^\sigma}\, ds, \quad \beta < 1 \tag{3'}$$

3. $\beta = 0$.

This case was most frequently investigated [3—8, 10, 12, 13]. In reality it was the function:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{ts - s^\sigma}\, ds, \quad 0 < t < \infty,$$

which was being considered.

Besides the relation (5) for $\beta = 0$ the following relations were also proved [6]:

$$x^{-1}\Phi(0, -\sigma; -x^{-\sigma}) = \frac{2}{\pi}\int_0^\infty \exp\left(-\cos\frac{\pi\sigma}{2} t^\sigma\right) \sin\left(\sin\frac{\pi\sigma}{2} t^\sigma\right) \sin xt\, dt \tag{6}$$

$$x^{-1}\Phi(0, -\sigma; -x^{-\sigma}) = \frac{2}{\pi}\int_0^\infty \exp\left(-\cos\frac{\pi\sigma}{2} t^\sigma\right) \cos\left(\cos\frac{\pi\sigma}{2} t^\sigma\right) \cos xt\, dt \tag{7}$$

$$x^{-1}\Phi(0, -\sigma; -x^{-\sigma}) = \frac{1}{\pi}\frac{\sigma}{1-\sigma}\frac{1}{x}\int_0^\pi u e^{-u}\, dy \tag{8}$$

$$u = x^{-\frac{\sigma}{1-\sigma}}\left(\frac{\sin\sigma y}{\sin y}\right)^{\frac{\sigma}{1-\sigma}} \frac{\sin(1-\sigma)y}{\sin y}$$

P. Humbert [3] showed without proof and H. Pollard [7] proved that this function can be written in the form of a series:

$$-\frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)} \sin\pi\sigma k \frac{\Gamma(\sigma k + 1)}{x^{\sigma k+1}}, \quad x > 0.$$

But if one takes the known relation:

$$\Gamma(-\alpha k)\Gamma(1 + \alpha k)\sin(-\alpha k \pi) = \pi$$

one has the series (1) for $\beta = 0$ and $z = -x^{-\sigma}$.

For $\beta = 0$ and $\sigma = \frac{1}{2}$:

$$t^{-1}\Phi\left(0, -\frac{1}{2}; -x^{-\frac{1}{2}}\right) = \frac{1}{\pi}\int_0^\infty e^{-xu} \sin\sqrt{u}\, du = \frac{1}{2\sqrt{\pi t^3}} e^{-\frac{1}{4x}}. \tag{9}$$

Finally for $\beta=0$ and $\sigma=\frac{2}{3}$

$$t^{-1}\Phi\left(0,\ -\frac{2}{3};\ -x^{-\frac{2}{3}}\right)=\frac{-1}{2\sqrt{3\pi}\,x}\,e^{\frac{-2}{27x^2}}\,W_{-\frac{1}{2},\,-\frac{1}{6}}\left(-\frac{4}{27x}\right). \tag{10}$$

Where $W_{\mu,\nu}(x)$ is Whittaker's function which satisfies the differential equation:

$$\frac{d^2}{dx^2}W(x)+\left(-\frac{1}{4}+\frac{\mu}{x}+\frac{\nu^2}{4x^2}\right)W(x)=0.$$

3. *The integral relation for* Φ

Between the most important integral relations are these related to the Laplace transform. First, from (3) it follows:

$$\int_0^\infty e^{-sx}x^{\beta-1}\Phi(\beta,\ -\sigma;\ -x^{-\sigma}t)\,dx=s^{-\beta}e^{-ts^\sigma}\,. \tag{11}$$

If the function f has its Laplace transform $\varphi(s)$, $\mathscr{L}\{f(t)\}=\varphi(s)$, then [8]:

$$\mathscr{L}\left\{\int_0^\infty \Phi(0,\ -\sigma;\ -tx^{-\sigma})f(t)\frac{dt}{\sigma t}\right\}=s^{\sigma-1}\varphi(s^\sigma). \tag{12}$$

This formula is very useful when we try to find the function of which the Laplace transform is a given function.

It is also possible to give the corresponding relation for the two dimensional Laplace transform [9].

By definition:

$$\int\int_{(R+)^2} F(x,\ y)\,dx\,dy=\lim_{n\to\infty}\int\int_{P_n} F(x,\ y)\,dx\,dy$$

where P_n are rectangles: $0\leqslant x\leqslant x_n$, $0\leqslant y\leqslant y_n$ so that for every square Q from the first quadrant there exists a number n_0 such that $Q\subset P_n$ for every $n\geqslant n_0$.

For a function $h(\xi,\ \eta,\ x,\ y)$, defined in the domain $\Omega:\xi\geqslant\xi_1,\ \eta\geqslant\eta_1$, $x\geqslant 0$, $y\geqslant 0$, and for which there exists the integral:

$$\int\int_{(R+)^2} h(\xi,\ \eta,\ x,\ y)\,dx\,dy$$

we call the functions $H_i(\xi,\ \eta,\ x,\ y)$, $i=1,\ 2$, limiting functions if:

1. the following inequalities are satisfied:

$$H_1(\xi,\ \eta,\ x,\ y)\leqslant h(\xi,\ \eta,\ x,\ y)\leqslant H_2(\xi,\ \eta,\ x,\ y),\quad (\xi,\ \eta,\ x,\ y)\in\Omega$$

2. the functions:

$$H_i(\xi,\ \eta,\ x,\ y)\quad\text{and}\quad H_i(x,\ y)=\lim_{\xi,\,\eta\to\infty}H_i(\xi,\ \eta,\ x,\ y),\quad i=1,\ 2$$

are integrable on $(R^+)^2$ and:

$$\lim_{\xi, \eta \to \infty} \int\int_{(R+)^2} H_i(\xi, \eta, x, y)\, dx\, dy = \int\int_{(R+)^2} H_i(x, y)\, dx\, dy.$$

We shall also note with $P(u_0, u_1; v_0, v_1)$ the rectangle: $0 \leqslant u_0 \leqslant u \leqslant u_1$, $0 \leqslant v_0 \leqslant v \leqslant v_1$.

Theorem 1. *Let*

1. *$f(x, y)$ be a continuous function in all points of $(R^+)^2$*
2. *Integral*

$$\int\int_{(R+)^2} e^{-ux-vy} f(x, y)\, dx\, dy = \varphi(u, v)$$

converges in all points of the rectangle $P(u_0, u_1; v_0, v_1)$

3. *The function $e^{-ux-vy} g(x, y; \xi_0, \eta_0)$ where*

$$g(x, y; \xi_0, \eta_0) = \int_0^{\xi_0}\int_0^{\eta_0} \Phi(\mu, -\nu; -x^{-\nu})\, x^{\mu-1}\, \Phi(\beta, -\sigma; -\eta y^{-\sigma})\, y^{\beta-1} f(\xi, \eta)\, d\xi\, d\eta$$

has its limiting functions in all points (u, v) of the rectangle

$$P\left(u_1^{-\frac{1}{\nu}}, u_0^{-\frac{1}{\nu}}; v_1^{-\frac{1}{\sigma}}, v_0^{-\frac{1}{\sigma}}\right).$$

Then

$$\int\int_{(R+)^2} e^{-ux-vy} g(x, y; \infty, \infty)\, dx\, dy = \frac{1}{u^\mu v^\beta} \varphi(u^\nu, v^\sigma) \tag{13}$$

for every point of the rectangle $P\left(u_1^{-\frac{1}{\nu}}, u_0^{-\frac{1}{\nu}}; v_1^{-\frac{1}{\sigma}}, v_0^{-\frac{1}{\sigma}}\right)$.

Theorem 2. *We suppose that the conditions 1 and 2 of theorem 1 for the function $f(x, y)$ are satisfied.*

Let

$$g_1(x, y; \xi_0) = \int_0^{\xi_0} \Phi(\mu, -\nu, -\xi x^{-\nu})\, x^{\mu-1} f(\xi, y)\, d\xi$$

$$g_2(x, y; \eta_0) = \int_0^{\eta_0} \Phi(\beta, -\sigma; -y^{-\sigma})\, y^{\beta-1} f(x, \eta)\, d\eta$$

and the functions $e^{-ux-vy} g_1(x, y; \xi_0)$ and $e^{-ux-vy} g_2(x, y; \eta_0)$ have their limiting functions, the first for all points (u, v) of the rectangle $P\left(u_1^{-\frac{1}{\nu}}, u_0^{-\frac{1}{\nu}}; v_0, v_1\right)$ and the second for all points (u, v) of the rectangle $P\left(u_0, u_1; v_1^{-\frac{1}{\sigma}}, v_0^{-\frac{1}{\sigma}}\right)$.

Then

$$\int\int_{(R+)^2} e^{-ux-vy} g_1(x, y; \infty)\, dx\, dy = \frac{1}{u^\mu} \varphi(u^\nu, v) \tag{14}$$

$$(u, v) \in P\left(u_1^{-\frac{1}{\nu}}, u_0^{-\frac{1}{\nu}}; v_0, v_1\right)$$

$$(15)\qquad \int\int_{(R+)^2} e^{-ux-vy} g_2(x, y; \infty)\, dx\, dy = \frac{1}{v^{\beta}} \varphi(u, v^{\sigma})$$

$$(u, v) \in P\left(u_0, u_1; v_1^{-\frac{1}{\sigma}}, v_0^{-\frac{1}{\sigma}}\right).$$

In addition to these relations it is possible to give a number of others. We shall cite only two related to Laplace transform: From (5) it follows [8]:

$$(16)\qquad \mathscr{L}\left\{\frac{1}{\pi} x^{-\beta} e^{-x^{\sigma}\cos\sigma\pi} \sin(x^{\sigma}\sin\sigma\pi + \beta\pi)\right\} = s^{\beta-1}\Phi(\beta, -\sigma; -s^{-\sigma}), \qquad \beta < 1$$

where we used the theorem for the equality of two analitical functions.

Also:

$$(17)\qquad \mathscr{L}\left\{x^{\beta/2-1}\Phi\left(\beta, -\sigma; -x^{-\frac{\sigma}{2}}\right)\right\} = \frac{\sqrt{\pi}}{2^{\beta}} s^{-\frac{\beta}{2}} \Phi\left(\frac{\beta+1}{2}, -\frac{\sigma}{2}; -2^{\sigma} s^{\sigma/2}\right).$$

The known integral relations for the function Φ are:

$$(18)\qquad \int_0^{\infty} \Phi(0, -\sigma; -tx^{-\sigma})\,\Phi(\beta, \rho; -t^{\rho})\, t^{\beta-1}\frac{dt}{\sigma t} =$$

$$= x^{\sigma(\beta-1)}\Phi(\sigma(\beta-1)+1, \rho\sigma; -x^{\sigma\rho}), \quad \rho > 0 \quad \beta > 0.$$

The special case of this is:

$$(19)\qquad \int_0^{\infty} \Phi(0, -\sigma; -tx^{-\sigma})\, t^{\frac{\beta-1}{2}} J_{\beta-1}(2\sqrt{t})\frac{dt}{\sigma t} =$$

$$= x^{\sigma(\beta-1)}\Phi(\sigma(\beta-1)+1, \sigma; -x^{\sigma}), \quad \beta > 0.$$

Finally:

$$(20)\qquad \int_0^{\infty} \Phi(0, -v; -tx^{-v})\,\Phi(\beta, -\rho; -t^{-\rho})\, t^{\beta-1}\frac{dt}{\sigma t} =$$

$$= x^{\sigma(\beta-1)}\Phi(\sigma(\beta-1)+1, -\rho\sigma; -x^{-\sigma\rho}), \quad 0 < \rho < 1$$

$$(21)\qquad \int_0^{\infty} \Phi(0, -\sigma; -tx^{-\sigma})\, t^{\rho}\frac{dt}{\sigma t} = \frac{\Gamma(\rho+1)}{\Gamma(\sigma\rho+1)} x^{\sigma\rho}, \quad \rho > -1.$$

4. *The functional relation and the differential equation satisfied by* Φ

E. M. Wright [14] showed that the function Φ satisfies the following relations:

$$(22)\qquad \begin{aligned} -\sigma z\,\Phi(\beta-\sigma, -\sigma; z) &= \Phi(\beta-1, -\sigma; z) + (1-\beta)\Phi(\beta, -\sigma; z) \\ \frac{d}{dz}\Phi(\beta, -\sigma; z) &= \Phi(\beta-\sigma, -\sigma; z) \end{aligned}$$

from which we can derive:

$$(23)\qquad -\sigma z \frac{d}{dz}\Phi(\beta, -\sigma; z) = \Phi(\beta-1, -\sigma; z) + (1-\beta)\Phi(\beta, -\sigma; z).$$

We shall consider some other relations which were very useful in the definition of a family of semi-norms in some subspaces of Mikusiński's operators [2]. Let as usually $\mathscr{L}\{f(t)\}$ denote the Laplace transform of the function f. It is known that from

$$\mathscr{L}\{f(t)\}=g(z)$$

it follows for $a>0$:

$$\mathscr{L}\{f(at)\}=\frac{1}{a}\,g\left(\frac{z}{a}\right).$$

The relation (11) can be written:

$$\mathscr{L}\{t^{\beta-1}\Phi(\beta,\ -\sigma;\ -t^{-\sigma})\}=z^{-\beta}e^{-z^{\sigma}},$$

hence

$$\mathscr{L}\{t^{\beta-1}\Phi(\beta,\ -\sigma;\ -a^{-\sigma}t^{-\sigma})\}=z^{-\beta}e^{-\left(\frac{z}{a}\right)^{\sigma}}.$$

We shall introduce the substitution $a=1/(\alpha)^{1/\sigma}$, where $\alpha^{1/\sigma}$ is the principal branch of this function:

$$\mathscr{L}\{t^{\beta-1}\Phi(\beta,\ -\sigma;\ -\alpha t^{-\sigma})\}=z^{-\beta}e^{-\alpha z^{\sigma}}$$

and also:

$$\mathscr{L}\left\{t^{\frac{\beta}{k}-1}\Phi\left(\frac{\beta}{k},\ -\sigma;\ -\frac{\alpha}{k}t^{-\sigma}\right)\right\}=z^{-\frac{\beta}{k}}e^{-\frac{\alpha}{k}z^{\sigma}}$$

hence:

$$\mathscr{L}\{t^{\beta-1}\Phi(\beta,\ -\sigma;\ -\alpha t^{-\sigma})\}=\mathscr{L}^{k}\left\{t^{\frac{\beta}{k}-1}\Phi\left(\frac{\beta}{k},\ -\sigma;\ -\frac{\alpha}{k}t^{-\sigma}\right)\right\}.$$

Now we have:

(24) $$t^{\beta-1}\Phi(\beta,\ -\sigma;\ -\alpha t^{-\sigma})=\left[t^{\frac{\beta}{k}-1}\Phi\left(\frac{\beta}{k},\ -\sigma;\ -\frac{\alpha}{k}t^{-\sigma}\right)\right]^{*k}$$

where

$$[f(t)]^{*k}=\underbrace{f(t)*f(t)*\cdots*f(t)}_{k}$$

and

$$f(t)*g(t)=\int_{0}^{t}f(t-u)\,g(u)\,du.$$

With the same method we can derive:

(25) $$t^{\beta_1+\beta_2-1}\Phi(\beta_1+\beta_2,\ -\sigma;\ -(\alpha_1+\alpha_2)t^{-\sigma})=$$
$$=t^{\beta_1-1}\Phi(\beta_1,\ -\sigma;\ -\alpha_1 t^{-\sigma})*t^{\beta_2-1}\Phi(\beta_2,\ -\sigma;\ -\alpha_2 t^{-\sigma}).$$

When $\sigma=\frac{p}{q}$, p and q natural numbers, we can construct a differential equation which is satified by the function Φ [8].

From the second equation (22) it follows:

$$\frac{d^q}{dz}\Phi\left(\beta,\ -\frac{p}{q};\ z\right)=\Phi\left(\beta-p,\ -\frac{p}{q};\ z\right)$$

and from the third one:

$$z^{\frac{(\beta-1-p)q}{p}}\left(-\frac{p}{q}z^{1+\frac{p}{q}}\frac{d}{dz}\right)^p\left[z^{\frac{(1-\beta)q}{p}}\Phi\left(\beta,\ -\frac{p}{q};\ z\right)\right]=\Phi\left(\beta-p\ \ -\frac{p}{q};\ z\right),$$

hence

$$\frac{d^q}{dz}\Phi\left(\beta,\ -\frac{p}{q};\ z\right)=z^{\frac{(\beta-1-p)q}{p}}\left(-\frac{p}{q}z^{1+\frac{p}{q}}\frac{d}{dz}\right)^p\left[z^{\frac{(1-\beta)q}{p}}\times\right.$$
$$\left.\times\Phi\left(\beta,\ -\frac{p}{q};\ z\right)\right]. \tag{26}$$

5. *The asymptotic behaviour of* Φ

E. M. Wright gave the asymptotic expansion of the function Φ by the method of steepest descents [15]. His theorems are related to different domains of complex plane and different values of σ.

The most important is the first theorem which contains the case $z=-x$, x positive and this appears in our investigations.

We write $y=-z$, choose $\arg z$ and $\arg y$ to satisfy: $-\pi<\arg z\leqslant\pi$, $-\pi<\arg y\leqslant\pi$, and we write:

$$Y=(1-\sigma)(\sigma^\sigma y)^{\frac{1}{1-\sigma}}$$

$$Y_1=(1-\sigma)(\sigma^\sigma ze^{\pi i})^{\frac{1}{1-\sigma}},\qquad Y_2=(1-\sigma)(\sigma^\sigma ze^{-\pi i})^{\frac{1}{1-\sigma}}$$

hence:

$$Y_1=Y,\ -\pi<\arg z\leqslant 0 \text{ and } Y_2=Y,\ 0<\arg z\leqslant\pi.$$

L and M are any two positive integers; and K is a positive number (not always the same at each occurrence) independent of x, y, z and t, but depending on ε, σ, β, M and L. The constant implied in the 0-notation is of the type K. Finally, the constants $A_0, A_1, \dots, A_{M-1}$ are given by the well known asymptotic expansion of the Γ — function:

$$\frac{\Gamma(\sigma t+1-\beta)}{2\pi\sigma^{\sigma t}(1-\sigma)^{(1-\sigma)(t+1)}\Gamma(t+1)}=\frac{(-1)^m A_m}{\Gamma\left\{\left(1-\sigma\right)t+\beta+\frac{1}{2}+m\right\}}+$$
$$+0\left(\frac{1}{\Gamma\left\{(1-\sigma)t+\beta+\frac{1}{2}+M\right\}}\right),$$

where $\arg t$, $\arg\sigma t$ and $\arg(\sigma t+1-\beta)$ all lie between $-\pi$ and π.

We write $I(Y)$ for the "exponential" asymptotic expansion:

$$I(Y)=Y^{\frac{1}{2}-\beta}e^{-Y}\sum_{m=0}^{M-1}A_m Y^{-m}+0(Y^{-M}),$$

and $J(z)$ for the "algebraic" asymptotic expansion:

$$J(z)=\sum_{l=0}^{L-1}\frac{z^{\frac{\beta-1-l}{\sigma}}}{\sigma\Gamma(l+1)\Gamma\left\{1+\frac{\beta-l-1}{\sigma}\right\}}+0\left(z^{\frac{\beta-1-L}{\sigma}}\right).$$

Theorem 3. *If* $|\arg y| \leqslant \min\left(\frac{3}{2}\pi(1-\sigma),\ \pi\right) - \varepsilon$, *then*

(27) $$\Phi(\beta,\ -\sigma;\ z = I(Y).$$

Theorem 4. *If* $0 < \sigma < \frac{1}{3}$ *and* $|\arg z| \leqslant \pi(1-\sigma) - \varepsilon$ *then*:

$$\Phi(\beta,\ -\sigma;\ z) = I(Y_1) + I(Y_2).$$

Theorem 5. *If* $\sigma = \frac{1}{3}$ *and* $|\arg z| \leqslant \pi(1-\sigma) - \varepsilon$, *then*:

$$\Phi(\beta,\ -\sigma;\ z) = I(Y_1) + I(Y_2) + J(z).$$

Theorem 6. *If* $\frac{1}{3} < \sigma < 1$ *and* $|\arg z| \leqslant \frac{1}{2}\pi(3\sigma - 1) - \varepsilon$ *then*

$$\Phi(\beta,\ -\sigma;\ z) = J(z).$$

Theorem 7. *If* $\frac{1}{3} < \sigma < 1$, $|\arg y| \leqslant \pi$ *and* $\left|\arg z \pm \frac{1}{2}\pi(3\sigma - 1)\right| \leqslant$ $\leqslant \pi(1-\sigma) - \varepsilon$, *then* $\Phi(\beta,\ -\sigma;\ z) = I(Y) + J(z)$.

J. Mikusiński [6] considered the special case $\beta = 0$ and obtained

(28) $$x^{-1}\Phi(0,\ -\sigma;\ -x^{-\sigma}) \sim K x^{-\frac{2-\sigma}{2-2\sigma}} \exp\left(-A x^{-\frac{\sigma}{1-\sigma}}\right),\quad x \to 0^+$$

(29) $$x^{-1}\Phi(0,\ -\sigma;\ -x^{-\sigma}) \sim M x^{-1-\sigma},\quad x \to \infty$$

where

$$A = (1-\sigma)\,\sigma^{\frac{\sigma}{1-\sigma}},\qquad K = \frac{\sigma^{\frac{1}{2-2\sigma}}}{\sqrt{2\pi(1-\sigma)}} \quad\text{and}\quad M = \frac{\sin\sigma\pi}{\pi}\Gamma(1+\sigma).$$

It is easy to see that the relation (28) is a consequence of (27). The relation (29) can be read from the relation (5) using a theorem of Abel's type, which is showed in [8]. But the asymptotic behaviour of the function $x^{\beta-1}\Phi(\beta,\ -\sigma;\ -x^{-\sigma})$ when $x \to \infty$ can be obtained from (5) for all $\beta < 1$ and not only for $\beta = 0$.

We suppose first that $\beta = -k$, $k = 1, 2, \ldots$, then we have from (5):

$$t^{-k-1}\Phi(-k,\ -\sigma;\ t^{-\sigma}) = \frac{1}{\pi}\int_0^\infty e^{-tx}\, e^{-x^\sigma \cos\sigma\pi}\, x^k \times$$
$$\times (-1)^k \sin(x^\sigma \sin\sigma\pi)\,dx.$$

The behaviour of the subintegral function when $x \to 0^+$ is:

$$(-1)^k x^{k+\sigma}\frac{\sin\sigma\pi}{\pi}$$

By Abel's theorem [1] it follows:

$$t^{-k-1}\,\Phi(-k,\ -\sigma;\ -t^{-\sigma})\sim(-1)^k\frac{\Gamma(k+\sigma+1)\sin\sigma\pi}{\pi\, t^{k+\sigma+1}},\quad t\to\infty \tag{30}$$

$$k=1,\ 2,\ \ldots$$

Now let $\beta<1$ and $\beta\neq-k$, $k=0,\ 1,\ 2,\ \ldots$, then the behaviour of the subintegral function in (5) is:

$$\frac{\sin\beta\pi}{\pi}x^{-\beta},\quad x\to0^+$$

By the Abel's theorem;

$$t^{\beta-1}\,\Phi(\beta,\ -\sigma;\ -t^{-\sigma})\sim\frac{\sin\beta\pi}{\pi}\frac{\Gamma(1-\beta)}{t^{1-\beta}},\quad t\to\infty, \tag{31}$$

$$\beta<1,\quad \beta\neq-k,\quad k=0,\ 1,\ 2,\ \ldots$$

6. *The change of the sign and the zeros of the function* Φ

Already H. Pollard [7] has shown that $t^{-1}\Phi(0,\ -\sigma;\ -t^{-\sigma})$ is almost everywhere positive. L. Włodarski [13] obtained in reality the same result;

$$t^{-1}\,\Phi(0,\ -;\ -t)\geqslant0,\quad x>0$$

(because this function is continuous). But it is possible to get a more precise result [10]:

$$t^{-1}\,\Phi(0,\ -\sigma;\ -t^{-\sigma})>0,\quad x>0.$$

A more general proposition is the following [11]:

Theorem 8. *If* $\beta\geqslant0$, *then*

$$x^{\beta-1}\,\Phi(\beta,\ -\sigma;\ -x^{-\sigma})>0,\quad x>0.$$

But if $\beta<0$ *this function has at least one zero and there exists an interval in which this function is negative.*

7. *Majorants of the function* Φ

The inequality which is often used for this function can be derived from the relation (3′):

$$x^{-p-1}\,\Phi(-p,\ -\sigma;\ -x^{-\sigma})=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}s^p\exp(xs-s^\sigma)\,ds$$

which is given for $p>-1$.

$$\left|x^{-p-1}\,\Phi(-p,\ -\sigma;\ -x^{-\sigma})\right|\leqslant\frac{1}{2\pi}\int_{-i\infty}^{i\infty}\left|z\right|^p e^{-Re\, z^\sigma}\left|dz\right|$$

$$\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|y\right|^p e^{-|y|^\sigma\cos\frac{\sigma\pi}{2}}\left|dy\right|$$

$$\leqslant\frac{1}{\pi}\int_{0}^{\infty}y^n e^{-y^\sigma\cos\frac{\sigma\pi}{2}}\,dy$$

with the change $y^\sigma \cos\frac{\sigma\pi}{2}=x$ we get:

$$\leqslant \frac{1}{\sigma\pi} \frac{1}{\left(\cos\frac{\sigma\pi}{2}\right)^{\frac{p+1}{\sigma}}} \int_0^\infty x^{\frac{p+1}{\sigma}-1} e^{-x}\,dx$$

$$\leqslant \frac{1}{\sigma\pi} \frac{1}{\left(\cos\frac{\sigma\pi}{2}\right)^{\frac{p+1}{\sigma}}} \Gamma\left(\frac{p+1}{\sigma}\right),$$

The inequality obtained is:

$$|x^{-p-1}\Phi(-p,\ -\sigma;\ -x^{-\sigma})| \leqslant \frac{1}{\sigma\pi} \frac{1}{\left(\cos\frac{\sigma\pi}{2}\right)^{\frac{p+1}{\sigma}}} \Gamma\left(\frac{p+1}{\sigma}\right), \quad p>-1. \quad x\geqslant 0 \tag{32}$$

J. Mikusiński shoved in [5] the following majorant for the special case $p=0$:

$$x^{-1}\Phi(0,\ -\sigma;\ -\lambda x^{-\sigma}) < \left(\frac{\sigma\lambda}{t}\right)^{\frac{1}{1-\sigma}} \exp\left[-(1-\sigma)\,\sigma^{\frac{\sigma}{1-\sigma}}\left(\frac{\lambda}{t^\sigma}\right)^{\frac{1}{1-\sigma}}\right] \tag{33}$$

which is valid for $\frac{\lambda}{t^\sigma} > \frac{2}{\sigma^\sigma}$, $\lambda>0$.

We can see how good this majorant is if we take $\sigma=\frac{1}{2}$. In this case the relation (33) is:

$$\frac{\lambda}{2\sqrt{\pi t^3}} \exp\left(-\frac{\lambda^2}{4t}\right) < \left(\frac{\lambda}{2t}\right)^2 \exp\left(-\frac{\lambda^2}{4t}\right) \text{ for } \frac{\lambda}{\sqrt{t}} > 64.$$

REFERENCES

[1] G. Doetsch, *Handbuch der Laplace Transformation*, Band I, Basel (1950).

[2] O. Hadžić and B. Stanković, *Some theorems on fixed point in locally convex spaces*, Publ. Inst. Math. t. 10 (24), 1970, 9—19, Beograd

[3] P. Humbert, *Nouvelles correspondances symboliques*, Bull. Sc. Math. LXIX (1945), 121—129.

[4] J. Mikusiński, *Sur les fonctions exponentielles du calcul opératoire*, Studia Math. XII (1951), 208—224.

[5] J. Mikusiński, *Sur la croissance de la fonction opérationnelle exp* $(-s^\alpha\lambda)$, Bull. Ac. Pol. Sci. Cl. III, 4.7 (1956), 423—425.

[6] J. Mikusiński, *On the function whose Laplace transform is exp* $(-s^\alpha\lambda)$, $0<\alpha<1$, Studia Math. XVIII (1959), 191—198.

[7] H. Pollard, *The representation of exp* $(-x^\lambda)$ *as a Laplace integral*, Bull. Amer. Math. Soc. 52 (1946), 908—910.

[8] B. Stanković, *O jednoj klasi singu'arnih integralnih jednačina*, Zbornik radova SAN XLIII, kn. 4 (1955), 81—130.

[9] B. Stanković, *Abbildung gewisser Operationen durch die zweidimensionale Laplace-Transformation*, Publ. Inst. Math. Beograd, XI (1957), 1—8.

[10] B. Stanković, *Sur une fonction du calcul opérationnel*, Publ. Inst. Math. Beograd, VI (1954), 75—78.

[11] B. Stanković. *O nulama jedne funkcije operatorskog računa*, Godišnjak Filozofskog fakulteta Novi Sad, IV, (1959), 379—381.

[12] A. Wintner, *Cauchy's stable distributions and an explicit formula of Mellin* Amer. J. Math. 78 (1956), 819—861.

[13] L. Wlodarski, *Une remarque sur une classe de fonctions exponentielles du calcul opérationnel*, Studia Math. XIII (1953), 188—189.

[14] E. M. Wright *On the coefficients of power series having exponential singularities*, J. Lond. Math. Soc. 8 (1933), 71—79.

[15] E. M. Wright, *The generalized Bessel function of order greater than one*, Quart. J. Math. Oxford series, 2 (1940), 36—48.