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ON THE CONVERGENCE OF A LACUNARY FOURIER SERIES AND ITS CONJUGATE SERIES

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1. Let $f(x)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π.

Let

$$\Sigma (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \tag{1.1}$$

be a Fourier series of $f(x)$ with an infinity of gaps (n_k, n_{k+1}) such that $n_{k+1} - n_k \to \infty$.

The conjugate series of (1.1) is

$$\Sigma (b_{n_k} \cos n_k x - a_{n_k} \sin n_k x). \tag{1.2}$$

Let

$$\overline{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \frac{\psi(t)}{t\text{an} \frac{t}{2}} dt \tag{1.3}$$

where

$$\psi(t) = f(x+t) - f(x-t).$$

2. We discuss here the convergence of the series (1.1) and (1.2) when the function $f(x)$ satisfies certain conditions in a subinterval $|x - x_0| \leqslant \delta$ of, $(-\pi, \pi)$. Let I denote such a subinterval. We prove the following theorems.

Theorem 1. *If* (1.1) *is a Fourier series of* $f(x)$ *where* $f(x) \in L^2(I)$ *then* (1.1) *and* (1.2) *are almost everywhere convergent.*

Theorem 2. *If* (1.1) *is a Fourier series of* $f(x)$ *where* $f(x)$ *is of bounded variation in some subinterval* I, *then* (1.1) *is convergent to* $[f(x+0) + f(x-0)]/2$ *at any point where this expression has a meaning and* (1.2) *is convergent to* (1.3) *whenever* (1.3) *exists.*

3. *Proof of Theorem* 1.

Lemma: *If* $f(x) \in L^2(I)$ *and if* (1.1) *is a Fourier series of* $f(x)$, *then* $f(x), \in L^2(-\pi, \pi)$.

This is a special case of a very general theorem due to Paley and Wiener ([1], theorem VLII').

We also require the following known theorem due to Carleson [2].

Theorem. *Fourier series of a function* $f(x) \in L^2(-\pi, \pi)$ *converges almost everywhere.*

Now, our proof is easy.

If $f(x) \varepsilon L^2(I)$, then $f(x) \varepsilon L^2(-\pi, \pi)$ by the above lemma and hence (1.1) converges almost everywhere by Carleson's theorem.

Also, by Riesz-Fischer theorem, (1.2) is a Fourier series of $\bar{f}(x) \varepsilon L^2(-$ $-\pi, \pi)$ whenever $f(x) \varepsilon L^2(-\pi, \pi)$ and hence (1.2) converges almost everywhere by Carlesons's theorem.

Proof of Theorem 2.

If S_n are the partial sums and σ_n are the arithmetic means for the series

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots,$$

$$S_n - \sigma_n = \frac{u_1 + 2u_2 + 3u_3 + \cdots + nu_n}{n+1}.$$

In case of a lacunary series, where in calculating Fejér's sums, it is necessary to replace the absent terms by zero, we have

$$S_{n_k} - \sigma_{n_k} = \frac{n_1 u_{n_1} + n 2 u_{n_2} + \cdots + n_k u_{n_k}}{n_k + 1}. \tag{3.1}$$

Now, we take

$$u_{n_k} = a_{n_k} \cos n_k x + b_{n_k} \sin n_k x$$

in case of the series (1.1) and

$$u_{n_k} = b_{n_k} \cos n_k x - a_{n_k} \sin n_k x$$

in case of the series (1.2)

Under the conditions of the theorem, we have by [3]:

$$a_{n_k} = 0(1/n_k),$$

$$b_{n_k} = 0(1/n_k)$$

$$\therefore \quad u_{n_k} = 0(1/n_k) \text{ and hence}$$

$$n_k u_{n_k} = 0(1).$$

Now, the number of terms in the numerator of the right hand side of (3.1) is k.

$$\therefore \quad |S_{n_k} - \sigma_{n_k}| < \frac{Ak}{n_k},$$

where A is an absolute constant.

Now, $k/n_k \to 0$ as $k \to \infty$, whenever $n_{k+1} - n_k \to \infty$ and hence

$$|S_{n_k} - \sigma_{n_k}| \to 0.$$

Now, it is known that the Fourier series (1.1) is summable $(c, 1)$ to $[f(x+0) + f(x-0)]/2$ for every value of x for which this expression has a meaning, i.e.

$$\sigma_{n_k} \to [f(x+0) + f(x-0)]/2.$$

Hence $S_{n_k} \to [f(x+0) + f(x-0)]/2$ for every value value of x for which this expression has a meaning.

It is also known that (1.2) is summable $(c, 1)$ to (1.3) at every value of x for which (1.3) exists and hence by the same argument as used above, (1.2) converges to (1.3) whenever (1.3) exists.

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