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ON LAGUERRE AND HERMITE POLYNOMIALS

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1. In [2], Carlitz has shown that the solution of the functional equation

$$(\cot\alpha\sin\beta)^n\ f_n(x\tan\alpha)$$

$$(1.1)\qquad =\sum_{r=0}^{n}\binom{n}{r}\left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^r(\cos\beta)^{n-r}f_{n-r}(x\tan\beta)$$

is given by

$$(1.2)\qquad f_n(\lambda)=\sum_{r=0}^{n}\binom{n}{r}(1-\lambda)^r\lambda^{n-r}c_{n-r},$$

where c_{n-r} are arbitrary constants.

This solution $f_n(\lambda)$ is given by the generating relation of the type

$$(1.3)\qquad \sum_{n=0}^{\infty}f_n(x)\frac{t^n}{n!}=e^t\sum_{n=0}^{\infty}d_n\frac{(xt)^n}{n!}=e^tF(xt)$$

where

$$F(u)=\sum_{n=0}^{\infty}d_n\frac{u^n}{n!}.$$

It follows that the functions generated by the generating functions of the form $e^tF(xt)$ may have a relation of the form (1.1).

Further, Jyoti Choudhary [5] has shown that the sequence of polynomials $F_n(x)$ given by the relation

$$(1.4)\qquad e^{xt}G\left[\frac{t^2}{4}(x^2-1)\right]=\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!}$$

where

$$G(u)=\sum_{n=0}^{\infty}g_nu^n,\ g_n\neq 0$$

will have a relation of the type

$$(1.5)\qquad F_n(x)=\left(\frac{1-x^2}{1-y^2}\right)^{n/2}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{x(1-y^2)^{1/2}-y(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\right]^kF_{n-k}(y),$$

which reduces to the form

$$(1.6)\qquad F_n(\cos\alpha)=\left(\frac{\sin\alpha}{\sin\beta}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^kF_{n-k}(\cos\beta).$$

The particular cases of $F_n(x)$ have been discussed in [9], [8], [1], [3] for Legendre, Associated Legendre and Ultraspherical polynomials.

Starting from [7, p. 238], further we shall show that the polynomials $g_n(x)$ defined by

$$\sum_{n=0}^{\infty} g_n(x)\, t^n = \Phi(t)\, f(x t) \tag{1.7}$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!},$$

and

$$\Phi(t) = \sum_{n=0}^{\infty} b_n t^n,$$

will have a relation analogous to (1.1) and (1.6), i.e.,

$$g_n(x) = \sum_{m=0}^{n} \left(\frac{x}{y}\right)^{n-m} c_m\, g_{n-m}(y) \tag{1.8}$$

where c_m is defined by (2.1). Taking particular cases of this, we shall obtain some results involving Laguerre and Hermite polynomials, few of them are already known.

In particular, we shall obtain

$$(\cot\alpha \sin\beta)^n L_n^{(\nu)}(\tan\alpha) = \sum_{m=0}^{n} \binom{\nu+n}{m} \left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^m (\cos\beta)^{n-m} L_{n-m}^{(\nu)}(\tan\beta), \tag{1.9}$$

and

$$(\cot\alpha \sin\beta)^{n/2} H_n(\sqrt{\tan\alpha}) = \sum_{m=0}^{[n/2]} \frac{n!}{m!\,(n-2m)!} \left[\frac{\sin(\alpha-\beta)}{\sin\alpha}\right]^m (\cos\beta)^{\frac{n}{2}-m} H_{n-2m}(\sqrt{\tan\beta}), \tag{1.10}$$

where $L_n^{(\nu)}(x)$ and $H_n(x)$ are Laguerre and Hermite polynomials respectively.

2. Consider, if

$$\frac{\Phi(t)}{\Phi\left(\frac{xt}{y}\right)} = \sum_{m=0}^{\infty} c_m t^m, \tag{2.1}$$

where c_m is a function of x and y, and $y \neq 0$, $\Phi\left(\frac{xt}{y}\right) \neq 0$,

then from (1.7), we have

$$\sum_{n=0}^{\infty} g_n(x)\, t^n = \Phi\left(\frac{xt}{y}\right) f\left(y \cdot \frac{xt}{y}\right) \times \frac{\Phi(t)}{\Phi\left(\frac{xt}{y}\right)}$$

$$= \sum_{n=0}^{\infty} g_n(y) \left(\frac{xt}{y}\right)^n \sum_{m=0}^{\infty} c_m t^m$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_m t^n \left(\frac{x}{y}\right)^{n-m} g_{n-m}(y).$$

Therefore, equating the coefficients of t^n, we obtain

$$g_n(x)=\sum_{m=0}^{n} c_m\left(\frac{x}{y}\right)^{n-m} g_{n-m}(y)$$

which is (1.8).

Thus we obtain the expansion of $g_n(x)$ in terms of a series of the same polynomials but of an arbitrary non-zero variable y.

Replacing x by λy, we get the multiplication formula for $g_n(x)$:

$$g_n(\lambda y)=\sum_{m=0}^{n} c_m \lambda^{n-m} g_{n-m}(y), \tag{2.2}$$

where c_m will contain λ and y.

3. In (1.7) if we take

$$\Phi(t)=e^t \text{ and } g_n(x)=\frac{\sigma_n(x)}{n!}, \text{ then}$$

$$\sum_{n=0}^{\infty} \sigma_n(x)\frac{t^n}{n!}=e^t f(xt), \tag{3.1}$$

so that, from (2.1)

$$c_m=\frac{\left(\frac{y-x}{y}\right)^m}{m!},$$

and, therefore, from (1.8) we obtain

$$\frac{\sigma_n(x)}{n!}=\sum_{m=0}^{n}\left(\frac{x}{y}\right)^{n-m}\frac{\left(\frac{y-x}{y}\right)^m}{m!}\frac{\sigma_{n-m}(y)}{(n-m)!}$$

or,

$$\sigma_n(x)=\left(\frac{x}{y}\right)^n\sum_{m=0}^{n}\binom{n}{m}\left(\frac{y-x}{x}\right)^m\sigma_{n-m}(y). \tag{3.2}$$

In particular, if we replace x by xy in (3.2), we obtain

$$\sigma_n(xy)=\sum_{m=0}^{n}\binom{n}{m}(1-x)^m x^{n-m}\sigma_{n-m}(y), \tag{3.3}$$

or, if we prefer

$$\sigma_n(xy)=\sum_{m=0}^{n}\binom{n}{m}(1-x)^{n-m}x^m\sigma_m(y). \tag{3.4}$$

This result which is particular case of (3.2) is given in [7, p. 239].

Again, in (3.2) replacing x by $\tan\alpha$ and y by $\tan\beta$ we obtain

$$(\cot\alpha\sin\beta)^n\sigma_n(\tan\alpha)=\sum_{m=0}^{n}\binom{n}{m}\left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^m\cos^{n-m}\beta\,\sigma_{n-m}(\tan\beta). \tag{3.5}$$

This result was given by Chatterjea [4, p. 243] for the polynomials $\Phi_n(x)$ defined by the generating function

$$e^t I_0(xt)=\sum_{n=0}^{\infty}\Phi_n(x)\frac{t^n}{n!}. \tag{3.6}$$

We note that $\Phi_n(x)$ are particular case of $\sigma_n(x)$ if in (3.1) we replace $f(xt)$ by a particular function $I_0(xt)$.

4. In [6, p. 192], a multiplication formula for the Laguerre polynomial is given in the form

$$L_n^{(\nu)}(\lambda y) = \sum_{m=0}^{n} \binom{\nu+n}{m} \lambda^{n-m} (1-\lambda)^m L_{n-m}^{(\nu)}(y). \tag{4.1}$$

Chatterjea [4, p. 244] has proved the following formula for simple Laguerre polynomial:

$$(\cot\alpha \sin\beta)^n L_n(\tan\alpha) = \sum_{m=0}^{n} \binom{n}{m} \left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^m \cos^{n-m}\beta \, L_{n-m}(\tan\beta), \tag{4.2}$$

and has deduced the result

$$(1+x)^n L_n\left(\sqrt{\frac{1-x}{1+x}}\right) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} L_{n-m}\left(\frac{\sqrt{1-x^2}}{x}\right). \tag{4.3}$$

In this section, we shall obtain as a particular case of (1.8), more general results for the Laguerre polynomials.

In (1.7), setting

$$\Phi(t) = e^t \text{ and } f(xt) = {}_0F_1(-;\ 1+\nu;\ -xt),$$

and comparing with the well known generating function for the Laguerre polynomials [10, p. 201]

$$\sum_{n=0}^{\infty} \frac{L_n^{(\nu)}(x)\, t^n}{(1+\nu)_n} = e^t \, {}_0F_1(-;1+\nu;-xt) \tag{4.4}$$

we have

$$g_n(x) = \frac{L_n^{(\nu)}(x)}{(1+\nu)_n}$$

and from (2.1) it is clear that

$$c_m = \frac{\left(\frac{y-x}{y}\right)^m}{m!}.$$

Therefore, from (1.8) we obtain

$$L_n^{(\nu)}(x) = \sum_{m=0}^{n} \frac{\left(\frac{x}{y}\right)^n \left(\frac{y-x}{x}\right)^m (1+\nu)_n L_{n-m}^{(\nu)}(y)}{m!\,(1+\nu)_{n-m}}$$

$$= \left(\frac{x}{y}\right)^n \sum_{m=0}^{n} \binom{\nu+n}{m} \left(\frac{y-x}{x}\right)^m L_{n-m}^{(\nu)}(y). \tag{4.5}$$

From this it can be easily seen that on putting $x = \lambda y$ in (4.5) we get (4.1).

Next, on putting $x=\tan\alpha$ and $y=\tan\beta$ in (4.5) we obtain

$$(4.6)\qquad (\cot\alpha\sin\beta)^n L_n^{(\nu)}(\tan\alpha)$$

$$=\sum_{m=0}^{n}\binom{\nu+n}{m}\left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^m \cos^{n-m}\beta\, L_{n-m}^{(\nu)}(\tan\beta).$$

In this, taking $\beta=2\alpha$, we get

$$(4.7)\qquad (2\cos^2\alpha)^n L_n^{(\nu)}(\tan\alpha)=\sum_{m=0}^{n}\binom{\nu+n}{m}(\cos 2\alpha)^{n-m} L_{n-m}^{(\nu)}(\tan 2\alpha).$$

Further using $\cos 2\alpha=x$, (4.7) can be put in the form

$$(4.8)\qquad (1+x)^n L_n^{(\nu)}\left(\sqrt{\frac{1-x}{1+x}}\right)=\sum_{m=0}^{n}\binom{\nu+n}{m}x^{n-m} L_{n-m}^{(\nu)}\left(\frac{\sqrt{1-x^2}}{x}\right).$$

The relations (4.2) and (4.3) can be obtained from (4.6) and (4.8) on simply putting $\nu=0$.

Again, on putting $x=\frac{ty}{1+t}$ in (4.5) we obtain

$$(4.9)\qquad L_n^{(\nu)}\left(\frac{ty}{1+t}\right)=(1+t)^{-n}\sum_{m=0}^{n}\binom{\nu+n}{n-m}t^m L_m^{(\nu)}(y).$$

As an application of this result, we can derive certain integrals involving Laguerre polynomials:

Using the orthogonal property of the Laguerre polynomials

$$(4.10)\qquad \int_0^\infty y^\nu e^{-y}[L_n^{(\nu)}(y)]^2\,dy=\frac{\Gamma(1+\nu+n)}{n!},\quad R_e(\nu)>-1$$

we obtain from (4.9)

$$(4.11)\qquad \int_0^\infty y^\nu e^{-y} L_m^{(\nu)}(y)\, L_n^{(\nu)}\left(\frac{ty}{1+t}\right)dy=\frac{\Gamma(1+\nu+n)}{m!\,(n-m)!}\cdot\frac{t^m}{(1+t)^n}.$$

Again combining the result [10. p. 212], for arbitrary c,

$$\frac{1}{(1-t)^c}\,{}_1F_1\left[\begin{matrix}c; & \frac{-yt}{1-t}\\ 1+\nu; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(c)_n L_n^{(\nu)}(y)\,t^n}{(1+\nu)_n}$$

with (4.9) and using the orthogonal property of Laguerre polynomials we have,

$$(4.12)\qquad \int_0^\infty y^\nu e^{-y}\,{}_1F_1\left[\begin{matrix}c; & \frac{-yt}{1-t}\\ 1+\nu; & \end{matrix}\right]L_n^{(\nu)}\left(\frac{yt}{1+t}\right)dy$$

$$=\frac{(1-t)^c}{(1+t)^n}\,\frac{\Gamma(\alpha+n+1)}{n!}\,{}_2F_1[-n,e;1+\nu;-t^2].$$

5. In this section we shall obtain similar relations for the Hermite polynomials $H_n(x)$ defined by the generating function

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}=e^{(2xt-t^2)}. \tag{5.1}$$

In (1.7), if we take

$$\Phi(t)=e^{-t^2},\ f(xt)=e^{2xt}$$

and compare with (5.1), we get

$$g_n(x)=\frac{H_n(x)}{n!},$$

and from (2.1)

$$c_{2m}=\frac{\left(\dfrac{x^2-y^2}{y^2}\right)^m}{m!},\quad c_{2m+1}=0,$$

and hence, from (1.8) we get

$$\frac{H_n(x)}{n!}=\sum_{m=0}^{[n/2]}\left(\frac{x}{y}\right)^{n-2m}c_{2m}\frac{H_{n-2m}(y)}{(n-2m)!}$$

or,

$$H_n(x)=\left(\frac{x}{y}\right)^n\sum_{m=0}^{[n/2]}\frac{n!}{m!\,(n-2m)!}\left(\frac{x^2-y^2}{x^2}\right)^m H_{n-2m}(y). \tag{5.2}$$

Replacing x by λy we obtain

$$H_n(\lambda y)=\sum_{m=0}^{[n/2]}\frac{n!}{m!\,(n-2m)!}(\lambda^2-1)^m\lambda^{n-2m}H_{n-2m}(y). \tag{5.3}$$

In (5.2), on putting $y=1$, we get

$$H_n(x)=\sum_{m=0}^{[n/2]}\frac{n!}{m!\,(n-2m)!}(x^2-1)^m x^{n-2m}H_{n-2m}(1), \tag{5.4}$$

which is given in [10, p. 199].

Again, in (5.2) replacing x by $\sqrt{\tan\alpha}$ and y by $\sqrt{\tan\beta}$, we get

$$(\cot\alpha\sin\beta)^{n/2}H_n(\sqrt{\tan\alpha})$$

$$=\sum_{m=0}^{[n/2]}\frac{n!}{m!\,(n-2m)!}\left[\frac{\sin(\alpha-\beta)}{\sin\alpha}\right]^m(\cos\beta)^{\frac{n}{2}-m}H_{n-2m}(\sqrt{\tan\beta}).$$

which is (1.10).

On putting $\beta=2\alpha$ we get

$$(2\cos^2\alpha)^{n/2}H_n(\sqrt{\tan\alpha})=\sum_{m=0}^{[n/2]}\frac{(-1)^m n!}{m!\,(n-2m)!}(\cos 2\alpha)^{\frac{n}{2}-m}H_{n-2m}(\sqrt{\tan 2\alpha}). \tag{5.5}$$

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