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ON THE POLYNOMIALS OF TRUESDEL TYPE

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1. Introduction: Bell polynomials are defined as [1]

$$H_n(g, h) = (-1)^n e^{-hg} D^n e^{hg}; \quad D \equiv d/dx, \tag{1.1}$$

where h is a constant and g is some specified function. In this if we take $h=1$ and $g(x) = \alpha \log x - px^r$, we get the generalised Hermite functions [4]

$$H_n^r(x, \alpha, p) = (-1)^n x^{-\alpha} e^{px^r} D^n (x^\alpha e^{-px^r}).$$

Truesdel polynomials are defined as [2],

$$T_n^\alpha(x) = x^{-\alpha} e^x \left(x \frac{d}{dx}\right)^n [x^\alpha e^{-x}];$$

and R. P. Singh [3] has given the generalisation of the Truesdel polynomials as

$$T_n^\alpha(x, r, p) = x^{-\alpha} e^{px^r} \left(x \frac{d}{dx}\right)^n [x^\alpha e^{-px^r}].$$

Now let us define

$$G_n(h, g) = e^{-hg} \left(x \frac{d}{dx}\right)^n e^{hg} \tag{1.2}$$

which is derived from (1.1) by replacing d/dx by xd/dx.

2. We give some familiar properties of $\delta \equiv x \dfrac{d}{dx}$:

$$\begin{cases} \delta^n x^\alpha = \alpha^n x^\alpha, \\ \delta^n (U \cdot V) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \delta^{n-k} U \cdot \delta^k V, \\ F(\delta)\{e^{g(x)} \cdot f(x)\} = e^{g(x)} F(\delta + xg') f(x); \quad g' \equiv Dg(x), \\ e^{t\delta} f(x) = f(x e^t). \end{cases} \tag{2.1}$$

Now using (2.1) we see that

$$G_n(h, g) = [\delta + xhg']^n \cdot 1. \tag{2.2}$$

Here we immediately see that if $h = 1$, $g(x) = \alpha \log x - px^r$, then

$$G_n(h, g)) = T_n^\alpha(x, r, p)$$

which are the generalised Truesdel polynomials.
Now

$$e^{-hg}(xD)^n [e^{hg} f(x)] = e^{-hg} e^{hg} [\delta + xhg']^n \cdot f(x) = [\delta + xhg']^n \cdot f(x).$$

Let us assume

$$\mathcal{D} = \delta + xhg'; \tag{2.3}$$

then

$$e^{-hg}(xD)^n [e^{hg} f(x)] = \mathcal{D}^n f(x). \tag{2.4}$$

Thus

$$\mathcal{D}^n \cdot 1 = G_n(h, g).$$

Now we easily obtain

$$\mathcal{D}^n(U \cdot V) = \sum_{k=0}^{n} \binom{n}{k} \delta^k(U) \cdot \mathcal{D}^{n-k}(V); \tag{2.5}$$

if we put $V = 1$, then it is clear from (2.5)

$$\mathcal{D}^n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}^{n-k}(1) \cdot \delta^k$$

and hence

$$\mathcal{D}^n = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(h, g)\, \delta^k \tag{2.6}$$

and

$$\delta^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} G_{n-k}(h, g) \cdot \mathcal{D}^k. \tag{2.7}$$

In particular if we put $h = 1$, $g(x) = \alpha \log x - px^r$, then we have

$$\mathcal{D} = \delta + \alpha - rpx^r$$

which is R. P. Singh's operator [3].

Now again

$$\mathcal{D}^{n+k} = \mathcal{D}^n \cdot \mathcal{D}^k,$$

hence

$$G_{n+k}(h, g) = \mathcal{D}^n G_k(h, g) = \mathcal{D}^k G_n(h, g). \tag{2.8}$$

Thus with the help of (2.7) we obtain,

$$\delta^k G_n(h, g) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} G_{k-i}(h, g)\, G_{n+k}(h, g). \tag{2.9}$$

Now

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_n(h,g) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-hg} \delta^n \cdot e^{hg} = e^{-hg} \cdot e^{t\delta} \cdot e^{hg}$$

$$= e^{-hg} \exp[h \cdot g(x e^t)]; \text{ from (2.1).}$$

Thus the generating function for $G_n(h,g)$ is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_n(h,g) = \exp[h \cdot \{g(x e^t) - g(x)\}]. \tag{2.10}$$

Again with the help of (2.6), we get

$$e^{t\mathcal{D}} f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} G_n(h,g)\; e^{t\delta} \cdot f(x) \tag{2.11}$$

which can be written, with the help of (2.1) and (2.10), as

$$e^{t\mathcal{D}} \cdot f(x) = f(x e^t) \exp[h\{g(x e^t) - g(x)\}]; \tag{2.12}$$

when $f(x) = 1$

$$e^{t\mathcal{D}}(1) = \exp[h\{g(x e^t) - g(x)\}] \tag{2.13}$$

which is (2.10).

Also if we take $f(x) = G_m(h,g)$, then

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_{n+m}(h,g) = G_m[h, g(x e^t)] \times \exp[h\{g(x e^t) - g(x)\}]. \tag{2.14}$$

Now differentiating (2.10) with respect to t, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_{n+1}(h,g) = x h e^t g'(x e^t) \times \exp[h\{g(x e^t) - g(x)\}] \tag{2.15}$$

and differentiating with respect to x we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} D\, G_n(h,g) = h e^t g'(x e^t) \exp[h\{g(x e^t) - g(x)\}] \tag{2.16}$$

$$- h g'(x) \exp[h\{g(x e^t) - g(x)\}].$$

Now from (2.10), (2.15) and (2.16), we get the following recurrence relation

$$x D\, G_n(h,g) = G_{n+1}(h,g) - x h g'(x)\, G_n(h,g) \tag{2.17}$$

which reduces to

$$T^{\alpha}_{n+1}(x,r,p) = x D\, T^{\alpha}_n(x,r,p) + (\alpha - r p x^r)\, T^{\alpha}_n(x,r,p); \tag{2.18}$$

when $h = 1$, $g(x) = \alpha \log x - p x^r$.

This has been proved by the author [5].

Following results are also easily obtainable:

$$\sum_{n=0}^{\infty} G_n(h, g)\, t^n = [1 - t\mathcal{D}]^{-1} \cdot 1, \tag{2.19}$$

$$\sum_{k=0}^{n} \binom{n}{k} G_k(h, g)\, t^k = [1 + t\mathcal{D}]^n \cdot 1, \tag{2.20}$$

$$\sum_{k=0}^{n} \binom{n}{k} G_{n-k}(h, g)\, t^k = [t + \mathcal{D}]^n \cdot 1. \tag{2.21}$$

We have the following generalised rule of differentiation tor the operator δ as [3]

$$\delta_x^n f(z(x)) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{d^k}{dz^k} f(z) \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j}\, \delta_x^n z^j. \tag{2.22}$$

From this we get an expansion for $G_n(h, g)$ as

$$G_n(h, g) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} h^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} [g(x)]^{k-j}\, \delta_x^n [g(x)]^j. \tag{2.23}$$

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