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ON SPACES ASSOCIATED WITH INTEGRAL TRANSFORM

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Indroduction: According to Doetsch [1] the spaces associated with Integral Transforms are not Hilbert spaces. Recently it has been shown by Dutta and Ganguly [2 *A*] that the object space associated with Laplace Transforms is not necessarily even a linear metric space; In fact, functions are said to be Laplace transformable if they are finite in $[0, \infty]$ and belong to L, — space when multiplied by factors e^{-sx} and the functions then can be evaluated when the integral (i.e. transform) is known.

In the theory of Laplace Transforms the theory of convolution plays a very important basic role. Discussions on convolution require the integrability of product of the integrable functions which is equivalent to square-integrability.

Wiener [3] considered a theory of integral transforms as a theory of linear transformation of the L_2-space to a L_2-space. Recently a discussion on the abstract structure of the theory has been proposed by Dutta [4] revealing some interesting features and in that discussion for simplicity and convenience the object space and image space are taken to be Hilbert spaces. From this it follows that the study of class of functions associated with L_2- space are more useful.

Here, in this paper we shall investigate the set of functions f so that $e^{-sx}f$ for suitable s be finite in $(0, \infty)$ and square integrable in this same range and the value of the integral is known.

In the sequel the Laplace transformable functions under the new definition will be denoted by L.T.S.-set and after the introduction of topology by L.T.S. space.

Algebric structure

It is very easy to show that the L.T.S. set is a linear system [2 *B*]. But it is very interesting to note in this case that for any two functions $f_1, f_2 \in L.T.S$ — set the product function $f_1 f_2 \in L.T.S$ - set since $\int_0^\infty e^{-(s_1+s_2)n} f_1 f_2 \, dx$ exists when $\int_0^\infty e^{-s_1 x} f_1 \, dx$ and $\int_0^\infty e^{-s_2 x} f_2 \, dx$ both exist. The associative and commutative property being evident it is easy to deduce that the L.T.S.-set is a commutative Ring.

Topological structure

Definition. For a function f in the L.T.S. set the number s_f which is the g.l.b of all numbers s for which $e^{-sx}f$ is in L_2 is called the abcissa of convergence.

Note: 1. The existence of s_f for one of f can be established easily by using Dedekind-cut after Doetsch. For some functions f at this g.l.b., $e^{-s_f x} f$ remains in L_2: for others it is not.

Note: 2 We take $s_f \geqslant 0$ since for $s_f < 0$ the function itself is a member of the L_2-space and in this case s_f may be taken as Zero.

Compactification. We extend for our convenience in the subsequent discussion the L_2- space by adjoining an ideal point D where D denotes any function whose square integral is $\pm\infty$. Thus the one point compactification $L_2^* = L_2 + \{D\}$ with its topology defined by the basis consisting of:

i) all open sets of L_2
ii) all subsets U of L_2^* such that $L_2^* - U$ are closed and compact.

We now introduce a binary relation in L.T.S. set which is as follows: A binary relation R exists between two functions f and g if $fe^{-s_f x}$ and $ge^{-s_g x}$ are equal to a function $h(x) \in L_2^*$. R is evidently an equivalence relation and as usual it begets a partition into disjoint classes in the L.T.S. set. Every partition is indexed by a function $\in L_2$: obviously all these functions whose squares at their abcissas of convergence diverge to $\pm\infty$ are indexed by D. Hence the Class of partitions has one-to-one correspondence with L_2^*.

The coarsest topology of L.T.S.-Set S_{LT} consistent with the topology of L_2^* is found by the method of transference of topology [5]; the open sets in the L.T.S.-Sets are obtained from the inverse images of the open sets of L_2^* under the mapping $f: S_{LT} \to L_2^*$.

Now the topology being introduced, the L.T.S.-set is a L.T.S.-space. After the introduction of the coarsest topology we are now in search of a finer topology.

Metrization. Let f and g be two functions belonging to the L.T.S.-set. Let $\rho(f, g)$ be a functional defined over the Cartesian product of two L.T.S. -sets

$$\rho(f, g) = |S_f - S_g| + \sqrt{\int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|^2 \, dx} = \rho_1 + \rho_2$$

where S_f and S_g are the abcissas of convergence for f and g.

Note: We know that our space of real numbers (the distance functions) is locally compact. As it is sometimes done this space can be compactified by the introduction of our point $+\infty$.

This means that we shall admit that distance between two points may be infinite. In our discussion here we shall always take the compactified real space.

For our convenience with subsequent discussion we say that $f \in C$ if the square integral is convergent at the g.l.b. and $f \in D$ if it is otherwise.

Lemma 1 $\rho_1(f, g)$ *is evidently a pseudometric.*

Lemma 2 $\rho_2(f, g)$ *is a pseudometric.*

Proof. i) if $f=g$ evidently $\rho_2(f, g)=0$;

ii) Obviously, it is symmetric;

iii) for proving the triangle inequality we consider three functions f, g and h and associate along with them three indices s_f, s_g and s_h.

Case-I *Let* $f, g, h \in C$

The triangle inequality becomes very much evident.

Case II. When one of $f, g, h \in D$

a) Let $f \in D : \rho_2(f, g) = \infty : \rho_2(f, h) = \infty$: evidently $\rho_2(g, h)$ is finite. Then we have $\rho_2(f, g) = \rho_2(f, h) + \rho_2(g, h)$. As the distance function is symmetric, we need not consider the case when $g \in D$. Separately

b) ϑf $f, g \in C$ and $h \in D : \rho_2(f, g)$ is finite $\rho_2(f, h) = \infty : \rho_2(g, h) = \infty : \rho_2(f, g) < < \rho_2(f, h) + \rho_2(g, h)$.

Case III. Two of $f, g, h \in D$ and another $\in C$

a) $f, g \in D$ and $h \in C$

i) $f \sim g \in C : \rho_2(f, g)$ is then finite where as $\rho_2(g, h)$ and $\rho_2(f, h)$ are finite; then $\rho_2(f, g) < \rho_2(f, h) + \rho_2(g, h)$.

ii) $f \sim g \in D : \rho_2(f, g) = \infty : \rho_2(g, h) = \infty : \rho_2(f, h) = \infty \cdot \rho_2(f, g) = \rho_2(f, h) + \rho_2(g, h)$.

The other cases are similar and so the proofs are evident Class IV $f, g, h \in D$.

If $f \sim g \in C$, $f \sim h \in C$, $\rho_2(f, g)$, $\rho_2(f, h)$, $\rho_2(g, h)$ are all finite. This case is discussed in case I. The all other possible cases can be deduced from the discussion above.

Combining all these cases it is evident that the triangle inequality is valid for ρ_2. Hence the Lemma.

Theorem: ρ *is a metric.*

It is evident that ρ is a pseudo metric

$$\text{Besides if } \rho(f, g) = 0, \; |s_f - s_g| + \sqrt{\int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|^2 \, dx} = 0$$

The two portions being separately positive they must vanish separately i.e. $|s_f - s_g| = 0$ giving $s_f = s_g$ and

$$\sqrt{\int_0^\infty |e^{s_f x} f - e^{-s_g x} g|^2 \, dx} = 0 \quad \text{i.e.} \quad \int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|^2 \, dx = 0$$

$$\text{i.e.} \quad e^{-s_f x} f = e^{-s_g x} g.$$

But $s_f = s_g$. Hence $f = g$. (Here our mode of integration is Lebesgue-integration and as per convention functions which are equal almost everywhere are identified)

Hence the theorem and ρ is a metric. Hence L.T.S.-set is a metric space.

We now consider only the positive L.T.S.-Set and prove some theorems with the help of our metric.

Theorem I. *The additive operation is continuous in the positive L.T.S. -set.*

Let s_f, s_g, s_{f_n}, s_{g_n} be the abcissas of convergence of f, g, f_n and g_n respectively.

Case I. Let $s_g > s_f$: then there is a neighbourhood of s_g in which there is no element of the sequence s_{f_n} and as s_g is the limit of s_{g_n}, So this neighbourhood contains all s_{g_n} for $n \geqslant n_0$ and $s_{g_n} > s_{f_n}$ for $n \geqslant n_0$. Hence for all $n \geqslant n_0$ the abcissa of convergence of $f_n + g_n$ is s_{g_n}.

$$\rho(f_n + g_n, f+g) = |s_{g_n} - s_g| + \sqrt{\int_0^\infty |e^{-s_{g_n}x}(f_n+g_n) - e^{-s_g x}(f+g)|^2 dx}$$

$$\leqslant |S_{g_n} - S_g| + \sqrt{\int_0^\infty |e^{-S_{g_n}x} g_n - e^{-S_g x} g|^2 dx} + \sqrt{\int_0^\infty |e^{-S_{g_n}x} f_n - e^{-S_g x} f|^2 dx}$$

$$= \rho(g_n, g) + \sqrt{\int_0^\infty |e^{-S_{g_n}x + S_{f_n}x} e^{-S_{f_n}x} f_n - e^{-S_g x + S_f x} e^{-S_f fx}|^2 dx}$$

$$\leqslant \rho(g_n, g) + \sqrt{\int_0^\infty |e^{-S_{g_n}x + S_{f_n}x} \cdot e^{-S_{f_n}x} f_n - e^{-S_{g_n}x + S_{f_n}x} e^{-S_f x} f|^2 dx}$$

$$+ \sqrt{\int_0^\infty |e^{-S_{g_n}x + S_{f_n}x} \cdot e^{-S_f fx} - e^{-S_g x + S_f x} e^{-S_f x} f|^2 dx}$$

$$= \rho(g_n, g) + \sqrt{\int_0^\infty |e^{-S_{g_n}x + S_{f_n}x}|^2 |e^{-S_{f_n}x} f_n - e^{-S_f x} f|^2 dx}$$

$$+ \sqrt{\int_0^\infty |e^{-S_f x} f|^2 |e^{-S_{g_n}x + S_{f_n}x} - e^{-S_g x + S_f x}|^2 dx}$$

$$\leqslant \rho(g_n, g) + \rho(f_n, f) + \sqrt{\int_0^\infty |e^{-S_f x} f|^2 |e^{-S_{g_n}x + S_{f_n}x} - e^{-S_g x + S_f x}|^2 dx}$$

$\longrightarrow 0$ *as* $n \to \infty$.

Case II. $S_f = S_g$. In this case every neighbourhood of S_f and similarly in every neighbourhood of S_g there are elements of S_{g_n} and also S_{f_n}. But in some cases $S_{f_n+g_n} = S_{f_n}$ and in other cases S_{g_n}. But in both the cases we have.

$$\rho(f_n+g_n, f+g) = |S_{f_n} - S_f| + \sqrt{\int_0^\infty |e^{-S_{f_n}x}(f_n+g_n) - e^{-S_f x}(f+g)|^2 dx}$$

$$\text{or } |S_{g_n} - S_g| + \sqrt{\int_0^\infty |e^{-S_{g_n}x}(f_n+g_n) - e^{-S_g x}(f+g)|^2 dx}$$

$$\leqslant \begin{cases} \sqrt{\int_0^\infty |e^{-S_g x} g|^2 |e^{-S_{f_n}x+S_{g_n}x} - e^{-S_f x+S_g x}|^2 dx} \\ \sqrt{\int_0^\infty |e^{-S_f x}|^2 |e^{-S_{g_n}x+S_{f_n}x} - e^{-S_g x+S_f x}|^2 dx} \end{cases}$$

$$\to \{0 \ as \ n \to \infty \quad \text{i.e.} \quad 0(f_n+g_n) \to (f+g) \ as \, n \to \infty.$$

Hence the additive operation is continuous with respect to the given topology.

Note: The theorem is equally valid if some members of a sequence f_n or its limit belong to the set ∞.

Theorem 2. *The multiplication operation is continuous in the L.S.T. -set.*

$$\rho(f_n g_n, f_g) = |S_{f_n} + S_{g_n} - S_f - S_g| + \sqrt{\int_0^\infty |e^{-(S_{f_n}x+S_{g_n}x)} f_n g_n - e^{-(S_f+S_g)x} f_g|^2 dx}$$

$$\leqslant |S_{f_n} - S_f| + |S_{g_n} - S_g|$$

$$+ \sqrt{\int_0^\infty |e^{-S_{f_n}x - S_{g_n}x} f_n g_n - e^{-S_{f_n}x - S_{g_n}x} f_n g + e^{-S_{f_n}x - S_g x} f_n g - e^{-(S_f x + S_g x)} f g|^2 dx}$$

$$\leqslant |S_{f_n} - S_f| + |S_{g_n} - S_g| + \sqrt{\int_0^\infty |e^{-S_{f_n}x} f_n|^2 |e^{-S_{g_n}x} g_n - e^{-S_g x} g|^2 dx}$$

$$+ \sqrt{\int_0^\infty |e^{-S_g x} g|^2 |e^{-S_{f_n}x} f_n - e^{-S_f x} f|^2 dx} \to 0.$$

$$as \ f_n \to f, \ g_n \to g.$$

Hence the result.

The topology which we have obtained after introducing the new metric in the L. T. S.-set is consistent with the topology in the L_2^* which is only a sub-class of the space of our consideration. In fact, when the space is $L_2^*[0, \infty)$

the functions are square-integrable and the abcissas of convergence of the functions of our space become zero in that case,

$$\text{i.e.}\quad \rho(f, g) = |S_f - S_g| + \sqrt{\int_0^\infty |e^{-S_f x} f - e^{-S_g x} g|^2 dx}$$

$$= |0-0| + \sqrt{|e^{-0x} f - e^{-0x} g|^2 dn} = \sqrt{\int_0^\infty |f-g|^2 dx}$$

which is the common metric in L_2^*

Conclusion: The L. T. S.-set as has been metrized above is not a linear metric space since the theorem I is only true for positive L. T. S. -set. Counter examples can be easily constructed to show that for the whole of L. T. S.-set the continuity of additive operation is not valid. What we can say about the positive L. T. S. set is that it is a topological semi-group.

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