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ON THE ASYMPTOTIC BEHAVIOURS OF COSINE SERIES WITH MONOTONE COEFFICIENTS

Chi-Hsing Yong

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§ 1. Let $f(x)$ and $g(x)$ be defined by

$$f(x)=\sum_{1}^{\infty} a_n \cos nx, \tag{1—1}$$

$$g(x)=\sum_{1}^{\infty} a_n \sin nx, \tag{1—2}$$

whenever the series converge.

If $\{a_n\}$ is monotonically decreasing to zero, both (1—1) and (1—2) converge uniformly outside an arbitrarily small neighbourhood of $x=0$. Near $x=0$, these series do converge, but may be unbounded. The asymptotic behaviours of $f(x)$ and $g(x)$, as $x\to+0$, appear first to have been investigated by Haslam-Jones [4], and then by Hardy [2; 3]. Later several authors generalized their theorems. The following result is due to Aljančić, Bojanić and Tomić [1]:

Suppose that $0<\beta<2$, $A_1>0$* and $a_n \downarrow 0$. Then**

$$g(x)\simeq \frac{A_1 \pi}{2\Gamma(\beta)\sin\frac{1}{2}\beta\pi}\, x^{\beta-1} L\left(\frac{1}{x}\right) \qquad \text{as } x\to+0,$$

if, and only if, $a_n \simeq A_1 n^{-\beta} L(n)$ as $n\to\infty$. Here $L(x)$ denotes a function which is slowly varying in the sense defined by Karamata. Some properties of $L(x)$ will be indicated in Lemma 1.

The object of this paper is to establish a parallel theorem for $f(x)$:

Theorem. *Suppose that* $0<\beta<1$ *and* $a_n \downarrow 0$ *ultimately. Then*

$$f(x)\simeq \frac{\pi}{2\Gamma(\beta)\cos\frac{1}{2}\beta\pi}\, x^{\beta-1} L\left(\frac{1}{x}\right) \quad \text{as } x\to+0, \tag{1—3}$$

if and only if, $a_n \simeq n^{-\beta} L(n)$ *as* $n\to\infty$.

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* Here and afterwards A_i $(i=1, 2, \ldots, 5)$ denote positive constants.

** $f_1(x)\simeq f_2(x)$ as $x\to a$, means that $\frac{f_1(x)}{f_2(x)}\to 1$ as $x\to a$.

§ 2. Preliminary lemmas:

Definition. A function $L(x)$ is said to be "slowly varying in the sense of Karamata", if it is positive, continuous and

$$\frac{L(tx)}{L(x)} \to 1 \text{ as } x \to \infty \text{ for every fixed } t > 0 \text{ [5]}. \tag{2—1}$$

Lemma 1. *Suppose that $L(x)$ is slowly varying in the sense of Karamata.*

(i) If $\nu > 0$, then $x^{\nu} L(x) \to \infty$, $x^{-\nu} L(x) \to 0$ as $x \to \infty$,

and also $\max\limits_{0 \leq t \leq x} \{t^{\nu} L(t)\} \simeq x^{\nu} L(x)$, $\max\limits_{x \leq t < \infty} \{t^{-\nu} L(t)\} \simeq x^{-\nu} L(x)$ as $x \to \infty$.

(ii) *If $\eta > 0$ and $-1 < \nu < 2$, then*

$$\int_0^{\eta} x^{\nu} \{(1-r)^2 + x^2\}^{-2} L\left(\frac{1}{x}\right) dx = \{A_2(\nu) + o(1)\} (1-r)^{\nu-3} L\left(\frac{1}{1-r}\right)$$

as $r \to 1-0$,

$$\textit{where } A_2(\nu) = \begin{cases} \dfrac{1}{2} \sin \dfrac{\nu\pi}{2} \Gamma(\nu) \Gamma(2-\nu) & \textit{for } \nu \neq 0, \\ \dfrac{1}{4} \pi & \textit{for } \nu = 0. \end{cases}$$

(iii) *If $a_n \downarrow 0$ and $0 < \nu < 2$, and*

$$\sum_1^{\infty} k r^k a_k \simeq \Gamma(2-\nu)(1-r)^{\nu-2} L\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1-0,$$

then $a_n \simeq n^{-\nu} L(n)$ *as* $n \to \infty$.

Karamata proved (i) in [5], and (ii) and (iii) are given in [1: p. 108] and [1: p. 113].

Lemma 2. *If $0 < r < 1$ and $0 \leqslant x \leqslant \pi$, then*

(i) $$\left| \frac{1}{(1-2r\cos x + r^2)^2} - \frac{1}{\{(1-r)^2 + x^2\}^2} \right| \leqslant \left(\frac{\pi^2}{4r} + 2\right)(1-r)^{-2} + 4(1-r)^{-3}, \text{ and}$$

(ii) $$\left| \frac{\left(2 \sin \frac{1}{2} x\right)^2}{(1-2r\cos x + r^2)^2} - \frac{x^2}{\{(1-r)^2 + x^2\}^2} \right| \leqslant \left(1 + \frac{1}{r^2}\right) + 4\left(1 + \frac{1}{r}\right)(1-r)^{-1}.$$

Proof: Since $\left\{(1-r)^2 + 4r \sin^2 \frac{x}{2}\right\}^2 \{(1-r)^2 + x^2\}^2$ is at least as large as each of $2(1-r)^6 x^2$, $(1-r)^4 x^4$, $\frac{8r}{\pi^2}(1-r)^2 x^6$, and $\frac{16 r^2}{\pi^2} x^8$, we obtain that

$$\left|\frac{1}{(1-2r\cos x+r^2)^2}-\frac{1}{\{(1-r)^2+x^2\}^2}\right|=$$

$$=\frac{\left\{\left(x^2-4\sin^2\frac{x}{2}\right)+4(1-r)\sin^2\frac{x}{2}\right\}\left\{\left(x^2+4r\sin^2\frac{x}{2}\right)+2(1-r)^2\right\}}{\left\{(1-r)^2+4r\sin^2\frac{x}{2}\right\}^2\{(1-r)^2+x^2\}^2}$$

$$\leqslant\frac{\{x^4+(1-r)x^2\}\{(1+r)x^2+2(1-r)^2\}}{\left\{(1-r)^2+4r\sin^2\frac{x}{2}\right\}^2\{(1-r)^2+x^2\}^2}.$$

Also,

$$\left|\frac{\left(2\sin\frac{x}{2}\right)^2}{(1-2r\cos x+r^2)^2}-\frac{x^2}{\{(1-r)^2+x^2\}^2}\right|=$$

$$=\left|\frac{16x^2(1-r^2)\sin^4\frac{x}{2}+(1-r)^4\left(4\sin^2\frac{x}{2}-x^2\right)+4x^2\sin^2\frac{x}{2}\left\{2(1-r)^3+x^2-4\sin^2\frac{x}{2}\right\}}{\left\{(1-r)^2+4r\sin^2\frac{x}{2}\right\}^2\{(1-r)^2+x^2\}^2}\right|$$

$$\leqslant\frac{2(1-r)x^6+\frac{1}{12}(1-r)^4x^4+2(1-r)^3x^4+\frac{1}{12}x^8}{\left\{(1-r)^2+4r\sin^2\frac{x}{2}\right\}^2\{(1-r)^2+x^2\}^2}.$$

Hence result.

§ 3. *Proof of the theorem:*

We prove the necessity first. Suppose that (1—3) holds. By Lemma 1 (i), $f(x)$ is integrable $L(0,\pi)$ and the a_n are the Fourier coefficients of $f(x)$. Since

$$\int_0^\pi\sum_1^\infty|kr^k\cos kxf(x)|\,dx\leqslant\sum_1^\infty kr^k\int_0^\pi|f(x)|\,dx<\infty,$$

we may multiply both sides of the identity

$$\sum_1^\infty kr^k\cos kx=\frac{r(1-r)^2-r(1+r^2)\,2\sin^2\frac{x}{2}}{(1-2r\cos x+r^2)^2}\qquad(0<r<1)$$

by $\frac{2}{\pi}f(x)$ and integrate termwise over $(0,\pi)$. Thus we get

$$\sum_1^\infty kr^k a_k=\frac{2r}{\pi}\int_0^\pi\frac{(1-r)^2f(x)}{(1-2r\cos x+r^2)^2}dx-\frac{4r}{\pi}\int_0^\pi\frac{(1+r^2)\sin^2\frac{x}{2}f(x)}{(1-2r\cos x+r^2)^2}dx=J_1(r)-J_2(r).$$

Putting $f(x)=x^{\beta-1}L\left(\frac{1}{x}\right)\bar{f}(x)$ and $A_3(\beta)=\dfrac{\pi}{2\,\Gamma(\beta)\cos\frac{1}{2}\beta\pi}$, then $\bar{f}(x)\to A_3(\beta)$ as $x\to+0$, and also $\bar{f}(x)$ is bounded for the interval $(0,\pi)$. Hence

$$J_1(r)=\frac{2r}{\pi}(1-r)^2\int_0^{\pi}x^{\beta-1}\{(1-r)^2+x^2\}^{-2}L\left(\frac{1}{x}\right)\bar{f}(x)\,dx$$

$$+\frac{2r}{\pi}(1-r)^2\int_0^{\pi}x^{\beta-1}\varphi_1(x,r)L\left(\frac{1}{x}\right)\bar{f}(x)\,dx$$

$$=J_{1,1}(r)+J_{1,2}(r),\ \text{where}\ \varphi_1(x,r)=\frac{1}{(1-2r\cos x+r^2)^2}-\frac{1}{\{(1-r)^2+x^2\}^2}.$$

Also,

$$J_2(r)=\frac{r}{\pi}(1+r^2)\int_0^{\pi}x^{\beta+1}\{(1-r)^2+x^2\}^{-2}L\left(\frac{1}{x}\right)\bar{f}(x)\,dx$$

$$+\frac{r}{\pi}(1+r^2)\int_0^{\pi}x^{\beta-1}\varphi_2(x,r)L\left(\frac{1}{x}\right)\bar{f}(x)\,dx$$

$$=J_{2,1}(r)+J_{2,2}(r),\ \ \text{where}\ \ \varphi_2(x,r)=\frac{\left(2\sin\frac{x}{2}\right)^2}{(1-2r\cos x+r^2)^2}-\frac{x^2}{\{(1-r)^2+x^2\}^2}.$$

Since $\int_0^{\pi}x^{\beta-1}L\left(\frac{1}{x}\right)dx<\infty$ for $0<\beta<1$, then, by Lemma 2, we have

$$|J_{1,2}(r)|\leqslant\sup_{0<x<\pi}\{\bar{f}(x)\}\left\{\frac{2}{\pi}\left(\frac{\pi^2}{2}+2\right)+\frac{8}{\pi}(1-r)^{-1}\right\}\int_0^{\pi}x^{\beta-1}L\left(\frac{1}{x}\right)dx$$

$$=o(1)(1-r)^{\beta-2}L\left(\frac{1}{1-r}\right),$$

and

$$|J_{2,2}(r)|=o(1)(1-r)^{\beta-2}L\left(\frac{1}{1-r}\right),\quad \text{as}\ r\to1-0.$$

On the other hand, for any $\varepsilon>0$, there is a $\delta>0$ such that $|\bar{f}(x)-A_3(\beta)|<\varepsilon$ whenever $|x|<\delta$. Then

$$\left|J_{1,1}(r)-\frac{2r}{\pi}(1-r)^2\int_0^{\delta}\frac{x^{\beta-1}}{\{(1-r)^2+x^2\}^2}L\left(\frac{1}{x}\right)A_3(\beta)\,dx\right|$$

$$=\frac{2r}{\pi}(1-r)^2\left|\int_0^{\delta}\frac{x^{\beta-1}}{\{(1-r)^2+x^2\}^2}L\left(\frac{1}{x}\right)\{\bar{f}(x)-A_3(\beta)\}\,dx+\right.$$

$$\left.+\int_{\delta}^{\pi}\frac{x^{\beta-1}}{\{(1-r)^2+x^2\}^2}L\left(\frac{1}{x}\right)\bar{f}(x)\,dx\right|$$

$$<\frac{2r}{\pi}(1-r)^2\left\{\varepsilon\int_0^{\delta}\frac{x^{\beta-1}}{\{(1-r)^2+x^2\}^2}L\left(\frac{1}{x}\right)dx+A_4\int_{\delta}^{\pi}\frac{x^{\beta-1}}{\{(1-r)^2+x^2\}^2}L\left(\frac{1}{x}\right)dx\right\}$$

$$=o(1)\,A_2(\beta-1)(1-r)^{\beta-2}L\left(\frac{1}{1-r}\right),\ \text{using Lemma 1 (ii).}$$

Again by Lemma 1 (ii), we have

$$J_{1,1}(r)\simeq\frac{2}{\pi}A_3(\beta)\,A_2(\beta-1)(1-r)^{\beta-2}L\left(\frac{1}{1-r}\right),\ \text{as } r\to 1-0.$$

Similarly, we also have

$$J_{2,1}(r)\simeq\frac{2}{\pi}A_3(\beta)\,A_2(\beta+1)(1-r)^{\beta-2}L\left(\frac{1}{1-r}\right).$$

Since $\frac{2}{\pi}A_3(\beta)\{A_2(\beta-1)-A_2(\beta+1)\}$

$$=\frac{1}{2\Gamma(\beta)\cos\frac{1}{2}\beta\pi}\left\{\sin\frac{\beta-1}{2}\pi\,\Gamma(\beta-1)\,\Gamma(3-\beta)-\sin\frac{\beta+1}{2}\pi\,\Gamma(\beta+1)\,\Gamma(1-\beta)\right\}$$

$$=\Gamma(2-\beta),$$

we obtain that

$$\sum_1^{\infty}kr^k a_k=J_1(r)-J_2(r)$$

$$\simeq\Gamma(2-\beta)(1-r)^{\beta-2}L\left(\frac{1}{1-r}\right)\ \text{as}\ r\to 1-0.$$

Using Lemma 1 (iii), we have the final result

$$a_n\simeq n^{-\beta}L(n)\quad\text{as}\ n\to\infty.$$

It is easier to prove the sufficiency. Suppose that $a_n\simeq n^{-\beta}L(n)$. Choose a small positive number δ and a large number η such that $0<\delta<1<\eta$, and put $p=\left[\frac{\delta}{x}\right]$, $q=\left[\frac{1}{x}\right]$, $r=\left[\frac{\eta}{x}\right]$. We may write

$$f(x)=\sum_{n=1}^{p}a_n\cos nx+\sum_{n=r+1}^{\infty}a_n\cos nx+\sum_{n=p+1}^{r}\{a_n-n^{-\beta}L(n)\}\cos nx$$

$$+\sum_{n=p+1}^{r}\{L(n)-L(q)\}\,n^{-\beta}\cos nx-L(q)\sum_{n=1}^{p}n^{-\beta}\cos nx$$

$$-L(q)\sum_{n=r+1}^{\infty} n^{-\beta}\cos nx + L(q)\sum_{n=1}^{\infty} n^{-\beta}\cos nx$$
$$=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6+\Sigma_7,$$

and investigate their order of magnitude as $x\to+0$. Firstly we have

$$\Sigma_7 \simeq \frac{\pi}{2\,\Gamma(\beta)\cos\frac{1}{2}\beta\pi}\,x^{\beta-1}\,L\left(\frac{1}{x}\right) \quad \text{as } x\to+0,$$

since $L(q)\simeq L\left(\frac{1}{x}\right)$ and $\sum_1^{\infty} n^{-\beta}\cos nx \simeq \frac{\pi}{2\,\Gamma(\beta)\cos\frac{1}{2}\beta\pi}\,x^{\beta-1}$ [6 : *p.* 186].

Secondly, it can be shown that all the sums other than Σ_7 are $o\left\{x^{\beta-1}L\left(\frac{1}{x}\right)\right\}$ as $x\to+0$. Put $a_n=n^{-\beta}L(n)\bar{a}_n$. Then $\bar{a}_n\to 1$ as $n\to\infty$, so $\{\bar{a}_n\}$ is bounded. By Lemma 1 (i) and for $\beta<\sigma<1$, we have

$$|\Sigma_1| \leqslant \sum_1^{p} n^{-\beta}\,L(n)\,\bar{a}_n < A_5 \max_{1\le n\le p}\{n^{\sigma-\beta}L(n)\}\int_0^{p} t^{-\sigma}\,dt$$
$$\simeq A_5\,\frac{1}{1-\sigma}\,p^{1-\beta}\,L(p) \text{ as } p\to\infty.$$

We also notice that $\frac{x^{1-\beta}}{L(q)}|\Sigma_1|\simeq A_5\frac{1}{1-\sigma}(xp)^{1-\beta}\simeq\frac{A_5}{1-\sigma}\delta^{1-\beta}$. Since δ can be chosen arbitrarily small, this proves that

$$\Sigma_1=o\left\{x^{\beta-1}L\left(\frac{1}{x}\right)\right\} \quad \text{as } x\to+0.$$

Also we have

$$|\Sigma_2|=\left|\sum_r^{\infty}\{a_n-a_{n+1}\}\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}-a_r\frac{\sin\left(r+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right| \leqslant \frac{2\pi}{x}a_r$$

$$\leqslant 2\pi\left\{(xr)^{-\beta}\frac{L(r)}{L(q)}\frac{a_r}{L(r)\,r^{-\beta}}\right\}x^{\beta-1}\,L(q)$$

$=o\left\{x^{\beta-1}L\left(\frac{1}{x}\right)\right\}$ as $x\to+0$, since η can be chosen arbitrarily large. It is obvious that $\Sigma_5=o\left\{x^{\beta-1}L\left(\frac{1}{x}\right)\right\}$, since

$$L(q)\sum_1^{p} n^{-\beta}\leqslant L(q)\int_0^{p} x^{-\beta}\,dx\simeq\frac{1}{1-\beta}L\left(\frac{1}{x}\right)x^{\beta-1}\,\delta^{1-\beta}$$
$$=o\left\{x^{\beta-1}L\left(\frac{1}{x}\right)\right\} \quad \text{as } x\to+0.$$

For Σ_3, Σ_4 and Σ_6, we may use the arguments given on pp. 111—112 of [1] by putting $A=1$.

The proof of the sufficiency is completed.

Postscript. The author is grateful to the referee who has pointed out that a paper by D. D. Adamović published recently (Publ. de l'Inst. Math., t. 7 (21), 1967, pp. 123—138, Theorem 1) contains our sufficiency part. (Zygmund has also proved a theorem similar to our sufficiency part, which covers less than the result proved here). Also by a theorem of Adamović (Matematički Vesnik 3 (18), 1966, pp. 161—172, Theorem IV) the slowly varying functions concerned in this paper can be easily extended to a more general class, viz., slowly varying functions in the following sense:

$L(x)$ is positive and measurable in an interval $0<x_0\leqslant x<\infty$, and $\dfrac{L(tx)}{L(x)}\to 1$ as $x\to\infty$ for every fixed $t>0$.

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Mathematics Department, Chung-Chi College,
The Chinese University of Hongkong, Shatin, N. T., Hongkong