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ON LAGUERRE POLYNOMIALS

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Let us consider the generating function [2, p. 202]

(1) $$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1+\alpha)_n} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\lambda} {}_1F_1\left((\lambda;\ 1+\alpha;\ -\frac{xt}{1-t}\right)$$

where $L_n^{(\alpha)}(x)$ is a Laguerre polynomial defined as [2, p. 200]

(2) $$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n;\ 1+\alpha;\ x).$$

In (1) we replace t by $t+u$, thus getting

$$\sum_{m,\,n=0}^{\infty} \binom{m+n}{m} \frac{(\lambda)_{m+n}}{(1+\alpha)_{m+n}} L_{m+n}^{(\alpha)}(x)\, t^n u^m$$

$$= (1-t)^{-\lambda} \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \varnothing_2\left(-m,\ \lambda+m;\ 1+\alpha;\ x,\ -\frac{xt}{1-t}\right) \frac{u^m}{(1-t)^m}.$$

Comparing the coefficients of u^m, we get

(3) $$\sum_{n=0}^{\infty} \binom{m+n}{m} \frac{(\lambda)_{m+n}}{(1+\alpha)_{m+n}} L_{m+n}^{(\alpha)}(x)\, t^n$$

$$= \frac{(\lambda)_m}{m!} (1-t)^{-\lambda-m} \varnothing_2\left(-m,\ \lambda+m;\ 1+\alpha;\ x,\ -\frac{xt}{1-t}\right),$$

where

$$\varnothing_2(\beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{m!\, n!\, (\gamma)_{m+n}} x^m y^n.$$

Putting $\lambda = 1 + \alpha$ and using the relation [2, p. 238]

$$\varnothing_2(\beta, \beta'; \beta + \beta'; x, y) = e^y {}_1F_1(\beta; \beta + \beta'; x - y),$$

(3) reduces to

$$(4) \qquad \sum_{n=0}^{\infty} \binom{m+n}{m} L_{m+n}^{(\alpha)}(x)\, t^n = e^{-\frac{xt}{1-t}} (1-t)^{-1-\alpha-m} L_m^{(\alpha)}\left(\frac{x}{1-t}\right).$$

(4) in formula (9) of [2, p. 211].

Now, by means of (3), we shall obtain some generating functions for Laguerre polynomials.

2. The generating functions that we propose to prove are

$$(5) \qquad \sum_{n=0}^{\infty} \frac{n!\, (\lambda)_n}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha)}(x)\, L_n^{(\beta)}(y)\, t^n$$
$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!\, (1+\alpha)_n (1+\beta)_n} \left(\frac{xyt}{1-t}\right)^n \psi_2\left(\lambda + n;\ 1 + \alpha + n,\ 1 + \beta + n;\ \frac{-xt}{1-t},\ \frac{-yt}{1-t}\right)$$

and

$$(6) \qquad \sum_{n=0}^{\infty} \frac{n!\, (m+n)!}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha)}(x)\, L_n^{(\beta)}(y)\, L_{m+n}^{(\gamma)}(z)\, t^n$$
$$= e^z (1-t)^{-1-\gamma-m} \sum_{n=0}^{\infty} \frac{(1+\gamma)_{m+n}}{n!\, (1+\alpha)_n (1+\beta)_n} \left(\frac{xyt}{1-t}\right)^n$$
$$\times \psi_2\left(1 + \gamma + m + n;\ 1 + \alpha + n,\ 1 + \beta + n,\ 1 + \gamma;\ -\frac{xt}{1-t},\ -\frac{yt}{1-t},\ -\frac{z}{1-t}\right),$$

where

$$\psi_2(\alpha; \beta', \beta'', \beta'''; x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha)_{m+n+p}}{m!\, n!\, p!\, (\beta')_m (\beta'')_n (\beta''')_p} x^m y^n z^p.$$

First, we give below the proof of (5).

Starting with the left-hand-side of (5) we have

$$\sum_{n=0}^{\infty}\frac{n!\,(\lambda)_n}{(1+\alpha)_n\,(1+\beta)_n}\,L_n^{(\alpha)}(x)\,L_n^{(\beta)}(y)\,t^n$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k\,(yt)^k}{(1+\beta)_k}\sum_{n=0}^{\infty}\binom{n+k}{k}\frac{(\lambda)_{n+k}}{(1+\alpha)_{n+k}}\,L_{n+k}^{(\alpha)}(x)\,t^n$$

$$=(1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!\,(1+\beta)_k}\,\varnothing_2\left(-k,\ \lambda+k;\ 1+\alpha;\ x,\ -\frac{xt}{1-t}\right)\left(\frac{yt}{1-t}\right)^k$$

by (3)

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!\,(1+\alpha)_n\,(1+\beta)_n}\left(\frac{xyt}{1-t}\right)^n\psi_2\left(\lambda+n;\ 1+\alpha+n,\ 1+\beta+n;\ \frac{-xt}{1-t},\ \frac{-yt}{1-t}\right),$$

which completes the proof of (5).

In (5) if we divide t by λ and let $\lambda\to\infty$, it becomes

$$\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n\,(1+\beta)_n}\,L_n^{(\alpha)}(x)\,L_n^{(\beta)}(y)\,t^n$$

(7)

$$=e^t\sum_{n=0}^{\infty}\frac{(xyt)^n}{n!\,(1+\alpha)_n\,(1+\beta)_n}\,{}_0F_1(-;\ 1+\alpha+n;\ -xt)\,{}_0F_1(-;\ 1+\beta+n;\ -yt).$$

With the help of the generating function for Jacobi polynomial [2, p. 251]

(8) $$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)\,t^n}{(1+\alpha)_n\,(1+\beta)_n}={}_0F_1\left[-;\ 1+\alpha;\ \frac{t(x-1)}{2}\right]{}_0F_1\left[-;\ 1+\beta;\ \frac{t(x+1)}{2}\right],$$

we write (7) as

$$e^{-t}\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n\,(1+\beta)_n}\,L_n^{(\alpha)}\left(\frac{1-x}{2}\right)L_n^{(\beta)}\left(-\frac{1+x}{2}\right)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(x^2-1)^n\,t^{n+k}}{n!\,2^{2n}\,(1+\alpha)_{n+k}\,(1+\beta)_{n+k}}\,P_k^{(\alpha+n,\,\beta+n)}(x),$$

which yields

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}\,k!}{(n-k)!\,(1+\alpha)_k\,(1+\beta)_k}\,L_k^{(\alpha)}\left(\frac{1-x}{2}\right)L_k^{(\beta)}\left(-\frac{1+x}{2}\right)$$

(9)

$$=\sum_{k=0}^{n}\frac{(x^2-1)^{n-k}}{2^{2n-2k}\,(n-k)!\,(1+\alpha)_n\,(1+\beta)_n}\,P_k^{(\alpha+n-k,\,\beta+n-k)}(x).$$

Again, using the generating function [2, p. 201]

(10)
$$\sum_{n=0}^{\infty} \frac{t^n}{(1+\alpha)_n} L_n^{(\alpha)}(x) = e^t\, {}_0F_1(-;\ 1+\alpha;\ -xt),$$

it follows that

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha)}(x)\, L_n^{(\beta)}(y)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{(xyt)^n}{n!\ (1+\alpha)_n (1+\beta)_n} e^{t/2}\, {}_0F_1(-;\ 1+\alpha+n;\ -xt)\, e^{t/2}\, {}_0F_1(-;\ 1+\beta+n;\ -yt)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(xyt)^n}{n!(1+\alpha)_{n+k} (1+\beta)_{n+r}} L_k^{(\alpha+n)}(2x)\, L_r^{(\beta+n)}(2y) \left(\frac{t}{2}\right)^{k+r},$$

from which we get

(11)
$$L_n^{(\alpha)}(x)\, L_n^{(\beta)}(y) = \sum_{k+r\leq n} \frac{(-n)_{k+r} (xy)^{n-k-r} (-\alpha-n)_r (-\beta-n)_k}{(n!)^2\, 2^{k+r}} \times L_k^{(\alpha+n-k-r)}(2x)\ L_r^{(\beta+n-k-r)}(2y).$$

Next, for proving (6) we replace in (7) t by ut and multiply both the sides by $u^{\lambda-1}\ e^{-u}\, {}_0F_1(-;\ 1+\gamma;\ uz)$, so that

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha)}(x)\, L_n^{(\beta)}(y)\, t^n u^{\lambda+n-1} e^{-u}\, {}_0F_1(-;\ 1+\gamma;\ uz)$$

$$= \sum_{n,k=0}^{\infty} \frac{(xyt)^n\, t^k}{n!\, k!\, (1+\alpha)_n (1+\beta)_n} u^{\lambda+n+k-1} e^{-u}\, {}_0F_1(-;\ 1+\alpha+n;\ -xut)$$

$$\times\, {}_0F_1(-;\ 1+\beta+n;\ -yut)\, {}_0F_1(-;\ 1+\gamma;\ uz).$$

Now integrating both the sides with respect to u with limits from 0 to ∞ and using

$$\Gamma(\lambda) = \int_0^{\infty} e^{-u} u^{\lambda-1}\, du,\ \ R(\lambda) > 0,$$

we arrive at

(12)
$$\sum_{n=0}^{\infty} \frac{n!\, (\lambda)_n}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha)}(x)\, L_n^{(\beta)}(y)\, {}_1F_1(\lambda+n;\ 1+\gamma;\ z)\, t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!\ (1+\alpha)_n (1+\beta)_n} \left(\frac{xyt}{1-t}\right)^n$$

$$\times\, \psi_2\!\left(\lambda+n;\ 1+\alpha+n,\ 1+\beta+n,\ 1+\gamma;\ \frac{-xt}{1-t},\ \frac{-yt}{1-t},\ \frac{z}{1-t}\right).$$

We put $\lambda = 1+\gamma+m$ and use the Kummer's formula

$$ {}_1F_1(\alpha;\ \beta;\ x) = e^x\, {}_1F_1(\beta-\alpha;\ \beta;\ -x). $$

Then using the definition (2), (12) leads to

$$ \sum_{n=0}^{\infty} \frac{n!\,(m+n)!}{(1+\alpha)_n\,(1+\beta)_n} L_n^{\alpha}(x)\, L_n^{(\beta)}(y)\, L_{m+n}^{(\gamma)}(z)\, t^n $$

(13)
$$ = e^z (1-t)^{-1-\gamma-m} \sum_{n=0}^{\infty} \frac{(1+\gamma)_{m+n}}{n!\,(1+\alpha)_n\,(1+\beta)_n} \left(\frac{xyt}{1-t}\right)^n $$

$$ \times \psi_2\left(1+\gamma+m+n;\ 1+\alpha+n,\ 1+\beta+n,\ 1+\gamma;\ \frac{-xt}{1-t},\ \frac{-yt}{1-t},\ \frac{-z}{1-t}\right). $$

This proves (6).

For $y=0$, (13) reduces to

(14)
$$ \sum_{n=0}^{\infty} \frac{(m+n)!}{(1+\alpha)_n} L_n^{(\alpha)}(x)\, L_{m+n}^{(\gamma)}(z)\, t^n = e^z (1-t)^{-1-\gamma-m} (1+\gamma)_m $$
$$ \times \psi_2\left(1+\gamma+m;\ 1+\alpha,\ 1+\gamma;\ \frac{-xt}{1-t},\ \frac{-z}{1-t}\right). $$

Putting $m=0$, $\gamma=\alpha$ and using the formula [1, p. 238]

$$ \psi_2(\alpha;\ \alpha,\ \alpha;\ x,\ y) = e^{x+y}\, {}_0F_1(-;\ \alpha;\ xy), $$

(14) yields the well-known Hardy-Hille formula

(15)
$$ \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y)\, t^n = (1-t)^{-1-\alpha} e^{-\frac{(x+y)t}{1-t}} $$
$$ \times {}_0F_1\left[-;\ 1+\alpha;\ \frac{xyt}{(1-t)^2}\right]. $$

3. In this section we obtain some more formulae involving Laguerre polynomials which are as follows:

(16) $$ \varnothing_1(\lambda;\ \beta;\ \mu;\ -xt,\ y) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} L_n^{(\alpha)}(x)\, F_1(\lambda+n\colon 1+\alpha+n,\ \beta;\ \mu+n;\ -t,\ y)\, t^n, $$

(17) $$ \psi_1(\lambda;\ \beta;\ \delta,\ \gamma;\ y,\ -xt) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\gamma)_n} L_n^{(\alpha)}(x)\, F_2(\lambda+n;\ 1+\alpha+n,\ \beta;\ \gamma+n,\ \delta;\ -t,\ y)\, t^n $$

$$\text{and } \Xi(\gamma,\ \beta;\ \delta;\ \lambda;\ y,\ -xt) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\lambda)_n} L_n^{(\alpha)}(x)$$

(18)

$$\times F_3(1+\alpha+n,\ \gamma;\ \beta+n,\ \delta;\ \lambda+n;\ -t,\ y)\, t^n.$$

The Appell functions F_1, F_2, F_3 and their confluent forms Φ, Ψ, Ξ, in (16), (17) and (18) are defined as [1, pp. 224—225]

$$F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_{m+n}} x^m y^n,$$

$$F_2(\alpha;\ \beta,\ \beta';\ \gamma,\ \gamma';\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_m(\gamma')_n} x^m y^n,$$

$$F_3(\alpha,\ \alpha';\ \beta,\ \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_{m+n}} x^m y^n,$$

$$\Phi_1(\alpha;\ \beta;\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_n}{m!\,n!\,(\gamma)_{m+n}} x^m y^n,$$

$$\Psi_1(\alpha;\ \beta;\ \gamma,\ \gamma';\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{m!\,n!\,(\gamma)_m(\gamma')_n} x^m y^n,$$

$$\Xi_1(\alpha,\ \alpha';\ \beta;\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m}{m!\,n!\,(\gamma)_{m+n}} x^m y^n.$$

The proof of (16), (17) and (18) depends upon [3, p. 207]

$$x^n = \sum_{k=0}^{n} \frac{(-1)^k\, n!\,(1+\alpha)_n L_k^{(\alpha)}(x)}{(n-k)!\,(1+\alpha)_k}.$$

We give the proof of (16) only, while others can be proved likewise. Thus we have

$$\varnothing_1(\lambda;\ \beta;\ \mu;\ -xt,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda)_{m+n}(\beta)_n}{m!\,n!\,(\mu)_{m+n}} (-xt)^m y^n$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m} \frac{(-1)^{m+k}(\lambda)_{m+n}(\beta)_n(1+\alpha)_m\, t^m y^n}{n!\,(m-k)!\,(\mu)_{m+n}(1+\alpha)_k} L_k^{(\alpha)}(x)$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{(\mu)_k} L_k^{(\alpha)}(x)\, F_1(\lambda+k;\ 1+\alpha+k,\ \beta;\ \mu+k;\ -t,\ y)\, t^k.$$

Similarly for (17) and (18).

4. Lastly, we are interested in proving the formulae which are

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y)\, L_m^{(\mu+n)}(xyt)$$

(20) $$= \frac{1}{m!} e^{xyt}(1-xt)^{\alpha}(1-yt)^{\beta} \sum_{n=0}^{\infty} \frac{(1+\mu)_{m+n}(-\alpha)_n(-\beta)_n}{n!\,(1+\mu)_n} \left[\frac{t}{(1-xt)(1-yt)}\right]^n$$

$$\times F_1\left(-m;\ -\beta+n,\ -\alpha+n;\ 1+\mu+n;\ \frac{xt}{xt-1},\ \frac{yt}{yt-1}\right),$$

$$L_{m_1}^{(\beta_1)}(x_1 t)\, L_{m_2}^{(\beta_2)}(x_2 t) \cdots L_{m_k}^{(\beta_k)}(x_k t)$$

$$= \frac{(1+\beta_1)_{m_1} \cdots (1+\beta_k)_{m_k}}{m_1! \cdots m_k!} e^{-t(1-x_1-\cdots-x_k)}$$

(21)

$$\times \sum_{n=0}^{\infty} \frac{\rho^n t^n}{n!} F_A\left(-n;\ -m_1,\ -m_2,\ \ldots,\ -m_k;\ 1+\beta_1,\ 1+\beta_2,\ \ldots,\ 1+\beta_k;\ -\frac{x_1}{\rho},\ -\frac{x_2}{\rho},\ \ldots,\ -\frac{x_k}{\rho}\right)$$

where $\rho = 1-x_1-x_2-\cdots-x_k$;

and F_A is defined as

$$F_A(\lambda;\ \alpha_1,\ \ldots,\ \alpha_n;\ \beta_1,\ \ldots,\ \beta_n;\ x_1,\ \ldots,\ x_n)$$

$$= \sum_{r_1,\ldots r_n=0}^{\infty} \frac{(\lambda)_{r_1+\cdots+r_n}(\alpha_1)_{r_1}\cdots(\alpha_n)_{r_n}}{r_1!\cdots r_n!\,(\beta_1)_{r_1}\cdots(\beta_n)_{r_n}} x_1^{r_1}\cdots x_n^{r_n}.$$

For proving the relation (20), we consider the formula due to Carlitz [1]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y) = e^{xyt}(1-xt)^{\beta}(1-yt)^{\alpha}$$

(22)

$$\times {}_2F_0\left[-\alpha,\ -\beta;\ -;\ \frac{t}{(1-xt)(1-ty)}\right]$$

or

$$e^{-xyt} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y) = \sum_{n,k,r=0}^{\infty} \frac{(-\beta)_{n+k}(-\alpha)_{n+r}}{n!\,k!\,r!} x^k y^k t^{n+k+r},$$

which on comparing the coefficients of t^n yields

$$\sum_{k=0}^{n} \binom{n}{k} L_{n-k}^{(\alpha-n+k)}(x)\, L_{n-k}^{(\beta-n+k)}(y)\, (-xy)^k$$

$$=(-\alpha)_n\, (-\beta)_n\, \psi_2(-n;\ 1+\alpha-n,\ 1+\beta-n;\ x,\ y)$$

or, if we prefer

(23)

$$\sum_{n=0}^{n} \binom{n}{k} L_{n-k}^{(\alpha+k)}(x)\, L_{n-k}^{(\beta+\mu)}(y)\, (-xy)^k$$

$$=(1+\alpha)_n\, (1+\beta)_n\, \psi_2(-n;\ 1+\alpha,\ 1+\beta;\ x,\ y).$$

Now, we notice that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!\,(\mu)_n} L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y)\, {}_1F_1(\lambda+n;\ \mu+n;\ -xyt)\, t^n$$

$$=\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} t^n \sum_{k=0}^{n} \frac{(-xy)^k}{k!\,(n-k)!} L_{n-k}^{(\alpha-n+k)}(x)\, L_{n-k}^{(\beta-n+k)}(y)$$

$$=\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} t^n \sum_{k+r\leqslant n} \frac{(-\alpha)_{n-k}\,(-\beta)_{n-r}}{k!\ r!\ (n-k-r)!} x^k y^r$$

$$=\sum_{n,\,k,\,r=0}^{\infty} \frac{(\lambda)_{n+k+r}\,(-\beta)_{n+k}\,(-\alpha)_{n+r}}{n!\ k!\ r!\ (\mu)_{n+k+r}} t^n (xt)^k (yt)^r$$

from which it follows that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!\,(\mu)_n} L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y)\, {}_1F_1(\lambda+n;\ \mu+n;\ -xyt)\, t^n$$

$$=\sum_{n=0}^{\infty} \frac{(\lambda)_n\,(-\alpha)_n\,(-\beta)_n}{n!\ (\mu)_n} t^n F_1(\lambda+n;\ -\beta+n,\ -\alpha+n;\ \mu+n;\ xt,\ yt).$$

We replace λ and μ by $1+\mu+m$ and $1+\mu$ respectively and then we use Kummer's transformation. This combined with (2) yields

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(\alpha-n)}(x) L_n^{(\beta-n)}(y) L_m^{(\mu+n)}(xyt)$$

(24)
$$= \frac{1}{m!} e^{xyt} \sum_{n=0}^{\infty} \frac{(1+\mu)_{m+n}(-\alpha)_n(-\beta)_n}{n!\,(1+\mu)_n} t^n$$

$$\times F_1(1+\mu+m+n;\ -\beta+n,\ -a+n;\ 1+\mu+n;\ xt,\ yt).$$

Now, if we use the relation [2, p. 239]

$$F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha;\ \beta,\ \beta';\ \gamma;\ \frac{x}{x-1},\ \ \frac{y}{y-1}\right),$$

(24) leads to (20).

Next, for proving (21) we observe that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} F_A(-n;\ \alpha_1, \alpha_2, \ldots, \alpha_k;\ \beta_1,\ \beta_2, \ldots, \beta_k;\ x_1,\ x_2, \ldots, x_k)$$

$$= \sum_{n=0}^{\infty} \sum_{r_1+\cdots+r_k \leq n}^{\infty} \frac{(\alpha_1)_{r_1}\cdots(\alpha_k)_{r_k}\, t^n (-x_1)^{r_1}\cdots(-x_k)^{r_k}}{r_1!\cdots r_k!\,(n-r_1-\cdots-r_k)!\,(\beta_1)_{r_1}\cdots(\beta_k)_{r_k}}$$

$$= \sum_{n, r_1, \ldots, r_k = 0}^{\infty} \frac{(\alpha_1)_{r_1}\cdots(\alpha_k)_{r_k}}{n!\, r_1!\cdots r_k!\,(\beta_1)_{r_1}\cdots(\beta_k)_{r_k}} (-x_1 t)^{r_1}(-x_2 t)^{r_2}\cdots(-x_k t)^{r_k}\, t^n,$$

which yields

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} F_A(-n;\ \alpha_1, \ldots, \alpha_k;\ \beta_1, \ldots, \beta_k;\ x_1, \ldots, x_k)$$

(25)
$$= e^t\, {}_1F_1(\alpha_1;\ \beta_1;\ -x_1 t)\cdots {}_1F_1(\alpha_k;\ \beta_k;\ -x_k t).$$

In (25) we replace α_i and β_i+m by $1+\beta_i+m$ and $1+\beta_i$, respectively, $i=1, 2, \ldots, k$, use Kummer's transformation. This with the help of (2) will lead us to

$$L_{m_1}^{(\beta_1)}(x_1 t)\cdots L_{m_k}^{(\beta_k)}(x_k\, t) = \frac{(1+\beta_1)_{m_1}\cdots(1+\beta_k)_{m_k}}{m_1!\cdots m_k!}$$

$$\times e^{-t(1-x_1-\cdots-x_k)} \sum_{n=0}^{\infty} \frac{t^n}{n!} F_A(-n;\ 1+\beta_1+m_1, \ldots 1+\beta_k+m_k;\ 1+\beta_1, \ldots, 1+\beta_k);$$

(26)
$$x_1,\ x_2, \ldots, x_k).$$

Now, if we use the relation

$$F_A(\lambda;\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_m;x_1,\ldots x_m)$$

$$=(1-x_1-\cdots-x_m)^{-\lambda}F_A\Big(\lambda;\beta_1-\alpha_1,\ldots,\beta_m-\alpha_m;$$

$$\frac{-x_1}{1-x_1-\cdots-x_m},\ldots,\frac{-x_m}{1-x_1-\cdots-x_m}\Big),$$

(26) assumes the form (21).

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