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SOME GENERATINGS AND PROPERTIES OF ORDERED SETS

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1. There are various ways of generating of ordered sets. The simplest way is certainly to consider any system S of sets ordered by the relation $\supset$ or by $\subset$. In particular, the power-sets PX (X being a set) yield the ordered sets $(PX, \supset)$ as the most general ordered sets, every ordered set being similar to a part of some $(PX, \supset)$. The cardinal ordering of any set $\{0,1\}^X$ furnishes the same possibility.

The procedure of intercalation, of inoculation, of hugging yields new ordered sets; it is then interesting how some properties of the obtained output set depend on similar properties of the given input sets.

In the paper we examine the foregoing procedures and examine the property of normality of ramified sets and of trees.

The ω_ν—ordinal dimension of A_ν—sets in the sense of Komm [5] is determined to be $k\,\omega_\nu$ and the problem is formulated as to whether there should be

$$d_0\,A_\nu < k\,\omega_\nu{}^{1)}$$

2. ***A sufficient condition for the normality of trees.***

2.1 *Degenerated ordered sets.* An ordered set $(0, \leqslant)$ is called degenerated, provided the comparability relation is transitive in the set $(0, \leqslant)$.

2.2. *Normal ordered sets.* An ordered set is called normal if it has the same cardinality as some of its degenerated subsets.

2.3. Theorem. *Let ω_ν be any regular non countable ordinal number. Every tree T of cardinality $\geqslant k\,\omega$ in which, for some $\omega'_\nu < \omega_\nu$, there is a strictly increasing function $f|T$ into a $k\,\omega'_\nu$-separable chain C contains an antichain of the cardinality $k\,\omega_\nu$.*

Proof. Let $r_0, r_1, r_n, (n < \omega'_\nu)$ be any simply ordered everywhere dense subset of C of cardinality $k\,\omega'_\nu$. For every $n < \omega'_\nu$ let T^n be the set of all the points of C satisfying

$$fx \leqslant r_n < f(b(a)),$$

1) kX = cardinality of X; $d_0 X$ is the minimal cardinal number n such that there is a system of n linear orderings of X, the superprosition of which produces the given ordering $(X, \leqslant)$.

where for any $a \in T$ we denote by $b(a)$ any successor of a (there is no restriction to assume that every $a \in T$ has infinitely many (even $k\,\omega_\nu$) successors). Obviously

$$T = \bigcup_n T^n \,(n < \omega_\nu') \text{ and}$$

$$kT = \Sigma_n\, kT^n \,(n < \omega_\nu').$$

The number kT being regular supposedly, there exists some index $m < \omega_\nu'$ such that $kT = kT^m$. Let us prove that T^m contains an antichain of cardinality $k\,\omega_\nu$. We can assume that the tree is of cardinality $k\,\omega_\nu$, that its every row as well as every chain are $< k\,\omega_\nu$, and consequently $\gamma\, T^m = \omega_\nu$. Moreover, one can suppose that every member a of T^m has $k\,\omega_\nu$ successors. By induction argument, (cf. [5] p. 486) one proves the existence of an ω_ν -sequence

$$a_\xi \in T^m \qquad (\xi < \tau = \omega_\nu)$$

such that the numbers ζ_ξ, defined by $a_\xi \in R_{\zeta_\xi} T$ form a strictly increasing ω_ν -sequence and that the numbers η_ξ defined by $b(a_\xi) \in R_{\eta_\xi} T$ satisfy

$$\zeta_\xi < \eta_\xi < \zeta_{\xi+1} \quad (\xi < \omega_\nu).$$

One proves then that the points $b(a_\xi)\,(\xi < \omega_\nu)$ form a requested antichain.

3. *A theorem on ramified sets.*

3.1. Definition. Any ordered set $(R, \leqslant)$ such that for every $x \in R$ the set $R(\cdot, x] \overset{\text{def}}{=} \{y \mid y \in R, y \leqslant x\}$ is *simply ordered*, is called a *ramified set*. Ramified sets generalize the trees. As an exercise one proves the following.

3.1.1. If D is a degenerated subset of a ramified set $(R, \leqslant)$, then the set

$$_0D(R) := \bigcup_x R(.\,, x] \quad (x \in D)$$

is a degenerated subset of $(R, \leqslant)$.

3.2. Theorem. *If a ramified set R is cofinal to a normal tree T, such that for every subset $S \subset T$ of cardinality cf kR one has*

$$(1) \qquad k \cup R[s] = kR, \; (s \in S),$$

then the set $(R \leqslant)$ is normal.

Proof. Since for every $x \in R$ the set $R(.\,, x]$ is simply ordered we might assume that

$$(2) \qquad kR(.\,, x] < kR \qquad (x \in R).$$

Now, by the definition of cofinality of R to T we have

$$(3) \qquad R = \bigcup_x R(.\,, x], \quad (x \in T).$$

3.2.1 First case: kR is regular. Then the relations (1), (2) jointly with the regularity of kR imply $kR = kT$. Now, let us consider the degenerated subset T_0 of T of the cardinality kT. By hypothesis, such a set exists. The number kT being regular, every set $T_0[a, \cdot)$ with $a \in R_0 T_0$ being a chain thus

$<kT$, we conclude that $kR_0T_0 = kT_0 = kR$. Now, for every $a \in R_0T_0$ let a' be any point of R such that $a \leqslant a'$; then the set $R'_0 := \{a' | a' \in R_0T_0\}$ is a requested antichain of R and obviously $kR'_0 = kR_0T_0 = kT_0 = kR$.

3.2.2. Second case: *kR is singular*. Of course, there is no restriction to assume that the cardinality of kT be a regular number $k\omega_\gamma$. The set T being normal, let then T_0 be a degenerated subset of T such that $kT_0 = kT$. Then we have two cases:

3.2.2.1. First case: The first row R_0T_0 of T_0 has kT_0 members. Since (1) and (2) hold for every $s \in R_0T_0$, we conclude easily that in every $R[s, \cdot)$, $(s \in R_0T_0)$ there exists a degenerated set of any cardinality $<kR$. Now, let us consider a well-ordering

$$t_0, t_1, \ldots, t_\alpha, \ldots \qquad (\alpha < \omega_\beta)$$

of the set R_0T_0 and any ω_β-sequence of cardinals k_α such that

$$k_\alpha < kR \text{ and } \Sigma_\alpha k_\alpha = kR, \ (\alpha < \omega_\beta). \tag{4}$$

In every $R[t_\alpha, \cdot), (\alpha < \omega_\beta)$ there exists some *degenerated* subset D_α of cardinality $\geqslant k_\alpha$; then the set $D = \bigcup_\alpha D_\alpha \, (\alpha < \omega_\beta)$ is a requested degenerated subset of R of cardinality $kR = \sum_\alpha k_\alpha \, (\alpha < \omega_\beta)$.

3.2.2.2. Second case: $kR_0T_0 < kT_0$. The number kT being regular, one concludes that for some $a \in R_0T_0$ we have

$$kT_0[a, \cdot) = kT_0. \tag{5}$$

Now, the set $T_0[a, \cdot)$ is well-ordered; T_0 being degenerated. Let us consider the chains

$$R(\cdot, x], \ (x \in T_0[a, \cdot)) \text{ and their union}$$

$$A \bigcup_x R(., x], \ (x \in T_0[a, \cdot)).$$

The set A is a simply ordered subset of $(R; \leqslant)$; if $kA = kR$, A is a requested subset of $(R \leqslant)$. Therefore, we have still to consider the case that $kA < kR$. By hypothesis (1) the relation (1) holds for $S = T_0[a, \cdot)$, i.e.

$$k \bigcup_S R(s) = kR \qquad (s \in T_0[a, \cdot)).$$

Consequently, there is a strictly increasing ω_β-sequence of points $a_\alpha \in T_0[a, \cdot)$ and a strictly increasing ω_β-sequence of cardinals k_α with $kA < k_\alpha < kR$ such that

$$kR[a_\alpha, \cdot) \geqslant k_{\alpha+1} \text{ and } \sum_\infty k_\alpha = kR, (\alpha < \omega_\beta).$$

We consider the sets

$$B_\alpha \overset{\text{def}}{=} R[a_\alpha,) \setminus R[a_{\alpha+1}, \cdot).$$

One has

$$\sup_\alpha kB_\alpha = kR.$$

Therefore, it is possible to choose degenerated sets

$$D_\alpha \subset B_\alpha$$

such that

$$\sup\ kD_\alpha = kR.$$

Now, the set

$$B = \bigcup_\alpha D_\alpha \qquad (\alpha < \omega_\beta)$$

is degenerated and has the cardinality kR, what completes the proof of the theorem.

3.3. *Transition: trees — ramified sets.* By intercalation of chains between consecutive members of a tree T one gets a ramified set; in other words, for any ordered pair (x, y) of consecutive members of a tree T let $c(x, y)$ be an ordered set — empty or non empty; if the sets $c(x, y)$ are pairwise disjoint and in no order relation, we intercalate $c(x, y)$ between the points x, y of T; if $x < y$, the set succeeds to every member of $T(., x]$ and preceeds to every member of $T(y, .)$. Let (T, c) be the ordered set so obtained.

Theorem. *The set (T, c) is ramified (a tree) if and only if for every $\{x, y\}_< \subset (T, \leqslant)$ the set $c(x, y)$ is a totally (well) ordered set.*

4. *Isomorph τ-dimension of ordered sets.*

4.1. *Definition of $i\tau$-dim.* Let Σ be the type of order of some linearly ordered set. If for some ordered set $(0, \leqslant)$ there exists a family F of linear extensions $(0, \leqslant_r)$ of $(0, \leqslant)$, each of order type τ, and such that for any $(a, b) \in 0^2$ one has $a \leqslant b$ if and only if $a \leqslant_r b$ for every $r \in F$, then the minimal cardinality kF of all such families F is called the isomorph τ-dimension of $(0, \leqslant)$ and is denoted $i\tau\text{-dim}\,(0, \leqslant)$.

4.1.1. E.g. for any finite ordered set $(0, \leqslant)$, if $k\,0 \equiv n$, then n-dim $(0, \leqslant)$ exists.

4.2. Theorem. *If*

(1) $(I\omega_0, \leqslant')$ *is any suborder of the linearly ordered set*

(2) $(I\omega_0, \leqslant)$,

then $i\omega_0$-dim $(I\omega_0; \leqslant')$ exists.

More generally we have the following.

4.3. Theorem. *In order that for some ordered set*

(1) $(0, \leqslant')$ *the $i\omega_0$-dim $(0, \leqslant')$ exists, it is necessary and sufficient that there is some one-to-one increasing mapping i of* (1) *into the chain*

(2) $(I\omega_0, \leqslant)$.

Obviously, the condition of the theorem is necessary. Let us prove that the condition of the theorem is also sufficient: if there exists some (1.1)-mapping i of (1) into (2), then there exists a family of ω_0-extensions of (1), the superposition of which yields the order (1).

For this purpose it is sufficient to prove that every antichain $\{a, b\}$ consisting of 2 incomparable points of (1) is obtainable by such ω_0-extensions of (1). Let $ia = o_1$, $ib = o_2$ and suppose $o_1 < o_2$. The antidomain $O' = iO$ of the set O is the union of the 2-point-set $\{o_1, o_2\}$, of the interval $O'(o_1, o_2)$, of the set $D' = O'(., o_1)$ and of the set $E' = O'(o_2, .)$. The set $I = O'(o_1, o_2)$ is the union of the following 3 subsets:

(3) $$A' = i\,0\,(a, \cdot)\bar{\leqslant}' \cap I, B' = iO\,(.\,,b)\leqslant' \cap I \text{ and } C' = i[CO\,[a]\leqslant' \cap CO\,[b]\leqslant' \cap I.$$

The domains of the corresponding subfunctions are well determined. Put $X = i^{-1} X'$, i.e.

(4) $$A = i^{-1} A', B = i^{-1} B', C = i^{-1} C', D = i^{-1} D', E = i^{-1} E'.$$

There is no restriction to suppose

(5) $$fa >= -fa + fb;$$

as a matter of fact, if this condition were not satisfied, we would consider the function $x \to f' x = fx + (-fa + fb) + 1$, and this function f' would satisfy the condition $f'a > -f'a + f'b$. This being so, let n be a member of $I\omega_0 := I$ such that

(6) $$o_1 > n > -o_1 + o_2;$$

we define a function $g|I$ in the following way:

(7) $$ga = fb = o_2, gb = o_1 = fa$$

$$g|A = -n + f|A,\ g|B = -n + f|B,\ g|C = f|C,\ g|D = -n + f|D,$$

$$g|E = n + g|E.$$

Let us define the ordering $(O; \leqslant_g)$ in such a way that for (x, y) we put

(8) $$x \leqslant_g y \Leftrightarrow gx \leqslant' gy.$$

The relation $\leqslant_g$ extends the relation $\leqslant'$ in $(O, \leqslant')$, i.e. for $(x, y) \in O^2$

(9) $$x <' y \Rightarrow gx <_g g_y, \text{ i.e. } g_x < gy.$$

The implication (9) is obvious if $\{x, y\}$ belongs to any of the sets

(10) $$A, B, C, D, E.$$

Therefore, we have to prove (9) if only one of the members of $\{x, y\}$ belongs to one of the sets (10), the another being in some another member of (10) or in $\{o_1, o_2\}$. E.g. assume $x \in A, y \in B$; then $gx = -n + fx$, $gy = -n + fy$. Since by hypothesis $x <' y$ so is $fx < fy$ and consequently

$$-n + fx < -n + fy, \text{ i.e. } gx < gy, \text{ i.e. } x <_g y.$$

In all other cases one proves (9) and also that the mapping $g|O$ is one-to-one. Therefore, $\leqslant_g$ is an order relation in S; in particular, we have $ga > gb$, and this jointly with $a < b$ proves that the superposition of the orderings $(O, \leqslant)$ and $(O, \leqslant_g)$ yields the incomparability of a, b in $(O; \leqslant')$.

4.4. Problem. Probably, in 4.2 and 4.3 it is legitimate to replace everywhere ω_0 by ω_α, for any ordinal α.

5. *On permutations of ordered sets.*

5.1. Let $(O, \leqslant)$ be any ordered set and O_1 the set of all the one-to-one mappings of O into itself. For any subset $F \subset O_1$ we define the order $\leqslant_F$ of O in the following way:

$$x \leqslant_F y \Leftrightarrow fx \leqslant fy \text{ for every } f \in F.$$

Obviously, $\leqslant_F$ is the total unorder for $F = 0_1$; for the identity transformation I of O the relation $\leqslant_{\{1\}}$ equals $\leqslant$.

5.2. Problem. *Is every suborder* $(O, \leqslant')$ of $(O, \leqslant)$ *obtainable as* $(O; \leqslant_F)$ *for some* $F \subset O_1$?

The answer is — yes! at least for $(I\,\omega_0, \leqslant)$ and probably for every $(I\,\omega_\alpha, \leqslant)$.

5.3. Theorem (*Superposition of* $< k\,\omega_\nu$ *orderings of* $I\,\omega_\nu$). *Let* ω_ν *be regular; any system* F *of cardinality* $< k\,\omega_\nu$ *of total* ω_ν*-orderings of the set* $I_\nu := I\,\omega_\nu = \{0, 1, 2, \ldots \alpha, \ldots\}_{\alpha < \omega_\nu}$ *yields by superposition an order* $(O, \leqslant)$ *of* I_ν *possessing an* ω_ν*-sequence in natural order.*

Proof. Let $F = \{f_\xi\}_\xi$ be a normal well order of F; let us define the ω_ν -sequence (1) a_ξ of numbers $< \omega_\nu$ in the following way: let $a_0 = O$; let a_1 be the first member of I_ν coming after a_0 in every member of F. Let $o < \beta < \omega_\nu$ and let suppose that the strictly increasing β-sequence $a_\alpha\,(\alpha < \beta)$ be defined; we define a_β as the first member of I_ν coming after $\{a_\alpha\}_\alpha$ in every member of F; since $k\,\beta < k\,\omega_\nu$ and since ω_ν is regular, the existence of a_β is guaranted. By induction arguments the sequence (1) of $k\,\omega_\nu$ points in strictly increasing order is defined. *Q. E. D.*

5.4. Theorem. *Let* ω_ν *be regular. The ordinal* ω_ν*-dimension of every* A_ν*-set exists and equals* $k\,\omega_\nu$ (*cf. Problem in 4.4.*).

As to the definition of A_ν-sets s. [5]. The existence of the ω_ν-dimension of A_ν follows from the fact that $(A_\nu, \leqslant_\nu)$ is obtainable by a family of permutations of $(I\,\omega_\nu, \leqslant)$ on the one hand and on the other hand of the fact that for any permutation p of the chain $I\,\omega_\nu$ the chain $(I\,\omega_\nu, \leqslant_p)$ is of the order type ω_ν. Finally, by 4.3. the ω_ν-dimension is not $< k\,\omega_\nu$ (cf. problem in 4.4).

5.6. Problem. *If a tree* $(T, \leqslant)$ *is the union of* $\leqslant k\,\omega_0$ *antichains, is then the ordinal dimension* $d_0 T$ *of the tree* $\leqslant k\,\omega_0$?

5.7. Problem. More generally, if an ordered set $(O, \leqslant)$ is the union of a family of $\leqslant b$ of its antichains, is then $d_0(O, \leqslant) \leqslant b$, i.e. is there a system F of cardinality $\leqslant b$ of total orderings of the set O such that $x \leqslant y$ in $(O, \leqslant)$ if and only if $x \leqslant_f y$ for every ordering $\leqslant_f \in F$.

6. *Hugging and inoculation of ordered sets.*

6.1. Let. $(O, \leqslant)$ be an ordered set. Let $f\,|\,O$ be any mapping such that to every point x of O one is associated a single ordered pair $(o_x, 1_x)$ of ordered sets. We denote by $O \otimes f$ the ordered set obtained from $(O, \leqslant)$ in such a way that O be extended by sets $o_x, 1_x$ and ordered in such a way that o_x precedes x, x precedes 1_x for every $x \in O$ and that o_x, 1_x be incomparable to $O(\cdot, x)$ and to $O(x, \cdot)$ respectively. In particular, o_x precedes 1_x in $O \otimes f$ for every $x \in O$.

6.1.1. The set $O \otimes f$ might be defined to consist of points of O and of ordered pairs (a, b) such that either $a \in O$ and $b \in 1_a$ or $b \in O$ and $a \in o_b$.

The ordering of $O \otimes f$ is performed in the following way: If $A, B \in O \otimes f$ then $A \leqslant B$ means exactly the following:

if $A, B \in O$, then $A \leqslant B$ in $(O, \leqslant)$;

if $A \in O$, and $B = (a, b)$, then $a \in O$, $A \leqslant a$ in $(O, \leqslant)$ and $b \in 1_a$;

if $B \in O$, and $A = (a, b)$, then $b \in O, b \leqslant B$ in $(O, \leqslant)$ and $a \in O_a$.

One verifies easily the following:

6.1.2. *Lemma.* For any ordered set $(O, \leqslant)$ and any mapping f defined in 6.1. the set $(O \otimes f, \leqslant)$ is an extension of $(O, \leqslant)$; in particular, for distinct points $x, y \in O$, the sets $0_x, 0_y$ are mutually incomparable as well as are the sets $1_x, 1_y$; one has $0_x \leqslant 1_y$ if and only if $x \leqslant y$; the sets $1_x, 0_y$ are incomparable mutually.

6.1.3. The processus $(O, \leqslant) f \to (O \otimes f; \leqslant)$ is called the hugging or hymerization (*cf* [2] p. 15) or the double inoculation of the sets fx to the set $(O, \leqslant)$.

6.2. *Inoculation.* If o_x equals $\varnothing$ identically, the hugging is called the *inoculation* or *grafting* in $(O, \leqslant)$. If $1_x = \varnothing$, the hugging of fx to x is called the *inverse inoculation* of the set 0_x at the point x of 0. If $f | O = X, Y)$ (constant), the f-hugging is denoted by $(X \leftarrow O \to Y)$; instead of $(\varnothing \leftarrow O \to Y)$ and $(X \leftarrow O \to \varnothing)$ we write also $(O \to Y)$, and $(X \leftarrow O)$ respectively.

6.2.1. Instead of $X \leftarrow (X \leftarrow O)$ we shall write $2X \leftarrow O$ in general, we define

$$(\alpha + 1) X \leftarrow O \text{ to be } X \leftarrow (\alpha X \leftarrow O) \text{ and}$$

$$\lambda X \leftarrow O := \cup_\beta \beta X \leftarrow O \quad (\beta < \lambda)$$

for any ordinal α and any limit ordinal λ.

Analogously, we put

$$(O \to Y) \to Y) = O \times 2\, Y$$

$$(O \to \alpha\, Y) = O \to (\alpha + 1)\, Y$$

$$O \to \lambda\, Y = \cup_\beta O \to \beta\, Y) \qquad (\beta < \lambda).$$

6.2.2. Convention. If (a, b, c) is any ordered triplet of ordered types, we define $a \leftarrow b \to c$ to mean $A \leftarrow B \to C$, where (A, B, C) is an ordered triplet of ordered sets of the ordered types a, b, c respectively.

6.2.3. Example. If (A, B) is an ordered pair of antichains and β any ordinal, then $T := A \to \beta\, B$ is a tree; the first row of T is A; the rank $\gamma = \gamma T$ of T is inf. $\{\beta + 1, \omega_0\}$; for every positive integer $n < \gamma T$ one has

$$kR_n T = (k\beta - n + 1)\, kA\, (kB)^n.$$

The proof of the last equality is performed by induction argument on n.

6.2.4. If $(O, \leqslant)$ as well as fx for every $x \in S$ are ramified (ranked) so is also the set $(O \otimes f; \leqslant)$; in particular, if $(O, \leqslant)$ and fx is a tree for every $x \in O$, then so is also the set $O \otimes f$.

6.3. Theorem. *Let $(O, \leqslant)$ be ordered and $x \in O \to fx = (0_x, 1_x)$ like in 6.1; then every regular ordinal number r which is representable in $O \otimes f$ is also representable in $(O, \leqslant)$ or in $0_x < x < 1_x$ for some $x \in 0$.*

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