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ON THE ABSOLUTE HARMONIC SUMMABILITY FACTORS OF A FOURIER SERIES*

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1. Let $\{s_n\}$ denote the n-th partial sum of a series Σa_n. The sequence--to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} \frac{1}{n-\nu+1} s_\nu, \tag{1.1}$$

where

$$P_n = \sum_{\nu=0}^{n} \frac{1}{\nu+1} \sim \log n,$$

defines the familiar Harmonic mean of the sequence $\{s_n\}$ [9]. The series Σa_n, or the sequence $\{s_n\}$, is said to be summable by Harmonic means, or summable $\left(N, \frac{1}{n+1}\right)$ to the sum s, if

$$\lim_{n\to\infty} t_n = s,$$

and is said to be absolutely Harmonic summable or summable $\left|N, \frac{1}{n+1}\right|$ to the sum s, if in addition the sequence $\{t_n\}$ is of bounded variation, that is to say,

$$\sum_n |t_n - t_{n-1}| < \infty.$$

It is known that the method is absolutely regular and implies absolute Cesàro summability of every positive order [5].

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$.

The Fourier series of $f(t)$ is

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t).$$

We write

$$\Phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}.$$

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2. Regarding the absolute Harmonic summability of a Fourier series the following theorems have been recently established.

Theorem A[7]. *There exists a function* $f(t)$ *of the class* (L) *such that* $\Phi(t)\log\frac{r}{t}$ $(r\geqslant 2\pi)$, *is of bounded variation in* $(0,\pi)$, *but its Fourier series, at* $t=x$, $\Sigma A_n(x)$ *is not summable* $\left|N, \frac{1}{n+1}\right|$.

Theorem B [11]. *If* $\Phi(t)$ *is of bounded variation in* $(0,\pi)$, *then the factored Fourier series* $\Sigma A_n(t)/\log(n+1)$ *is absolutely Harmonic summable.*

In Theorem 1 of the present paper which is one of a series of papers (See Lal [3]&[4]) devoted to the study of absolute Harmonic summability factors we determine suitable factors $\{\varepsilon_n\}$ in order that the associated Fourier series $\Sigma A_n(t)\varepsilon_n$ be absolutely Harmonic summable under the hypothesis of Theorem A. Theorem 2 is a generalisation of Theorem B. More precisely, we establish the following theorems:

Theorem 1. *If* $\Phi(t)\log\frac{r}{t}$ $(r>e^2\pi)$ *is of bounded variation in* $(0,\pi)$, *then the factored Fourier series* $\Sigma\log(n+1)\lambda_n A_n(t)$, *where* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma n^{-1}\lambda_n$ *is convergent, is absolutely Harmonic summable, at the point* $t=x$.

Theorem 2. *If* $\Phi(t)\left(\log\frac{r}{t}\right)^{1-\delta}$ $(0<\delta\leqslant 1)$ $(r>e^{2-\delta}\pi)$ *is of bounded variation in* $(0,\pi)$, *then the factored Fourier series* $\Sigma A_n(t)/\{\log(n+1\}^{\delta}$ *is absolutely Harmonic summable, at the point* $t=x$.

It is interesting to note that our hypothesis in Theorem 2 makes it possible to bridge the gulf between the hypotheses of Theorem 1 and Theorem B and provides us the suitable summability factor for the entire range of $0<\delta\leqslant 1$.

3. We require the following lemmas for the proof of our theorems.

Lemma 1. *If* $r>e^2\pi$, *the integral*

$$I\equiv\int_0^t\left(\log\frac{r}{u}\right)^{-2}\frac{\sin\mu u}{u}\,du=O\left[(\log r\mu)^{-2}\right]\quad(0<t\leqslant\pi).$$

Proof. Firstly we prove the lemma for the case $\mu t\geqslant 1$. Let us write I in the form

$$I=\left(\int_0^{\mu^{-1}}+\int_{\mu^{-1}}^{t}\right)\left(\log\frac{r}{u}\right)^{-2}\frac{\sin\mu u}{u}\,du\equiv I_1+I_2,\ \text{say.}\tag{3.1}$$

Since $\left(\log\frac{r}{u}\right)^{-2}$ is increasing in $(0,\mu^{-1})$, we have

$$I_1=(\log r\mu)^{-2}\int_{\eta}^{\mu^{-1}}\frac{\sin\mu u}{u}\,du=O\left[(\log r\mu)^{-2}\right].\quad(0<\eta<\mu^{-1}).\tag{3.2}$$

Also, since $u^{-1}\left(\log \frac{r}{u}\right)^{-2}$ is decreasing in (μ^{-1}, t), we have

(3.3) $$I_2=\mu(\log r\mu)^{-2}\int_{\mu^{-1}}^{\varsigma}\sin\mu u\,du=O\left[(\log r\mu)^{-2}\right],\qquad(\mu^{-1}<\varsigma<t).$$

The lemma for the case $\mu t\geqslant 1$, follows from (3.1), (3.2) and (3.3). Now we proceed to prove it for $\mu t<1$.

Here let us write I in the form

(3.4) $$I=\int_0^t\left(\log\frac{r}{u}\right)^{-2}\frac{\sin\mu u}{u}\,du$$

$$=\left(\int_0^{\pi}-\int_t^{\pi}\right)\left(\log\frac{r}{u}\right)^{-2}\frac{\sin\mu u}{u}\,du=I_3-I_4,\ \text{say}.$$

Proceeding on the same lines as in the previous case we can establish that

(3.5) $$I_3=O\left[(\log r\mu)^{-2}\right].$$

Again

(3.6) $$I_4=\left(\int_t^{\mu^{-1}}+\int_{\mu^{-1}}^{\pi}\right)\left(\log\frac{r}{u}\right)^{-2}\frac{\sin\mu u}{u}\,du=I_{4,1}+I_{4,2},\ \text{say}.$$

By arguments similar to those as in the cases of I_1 and I_2, it can be proved that

(3.7) $$\left.\begin{matrix}I_{4,1}\\ I_{4,2}\end{matrix}\right\}=O\left[(\log r\mu)^{-2}\right].$$

The lemma for the case $\mu t<1$ follows from (3.4), (3.5), (3.6) and (3.7). Thus the lemma is completely established.

Lemma 2. *If* $0<\delta<1$ *and* $r>e^{2-\delta}\pi$, *the integral*

$$\int_0^t\left(\log\frac{r}{u}\right)^{-2+\delta}\frac{\sin\mu u}{u}\,du=O\left[(\log r\mu)^{-2+\delta}\right].$$

Proof. The proof of this lemma is similar to that of Lemma 1.

Lemma 3 ([10] p. 440). *Uniformly for* $0<t<\pi$,

$$\left|\sum_{m}^{m'}\frac{\sin\nu t}{\nu}\right|=O(1),$$

for any positive integers m *and* m'.

L e m m a 4 [2]. *If* $0<t<\pi$, *then*

$$\left|\sum_{m}^{m'} \frac{\cos \nu t}{\nu}\right| = O\left[\log \frac{r}{t}\right], \quad r>\pi.$$

for all positive integers m and m'.

With the help of Lemmas 3 and 4 it is easy to deduce the following lemma.

L e m m a 5. *If* $0<t<\pi$, *then for all positive integers m and m'*

$$\left|\sum_{m}^{m'} \frac{\sin (n-\nu) t}{\nu}\right| = O\left[\log \frac{r}{t}\right], \quad (r>\pi).$$

L e m m a 6 ([1] Lemmas 3 and 4). *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma n^{-1}\lambda_n$ *is convergent, then* λ_n *is non-negative and decreasing,* $n\Delta\lambda_n = 0(1)$, *and* $\lambda_n \log n = 0(1)$, *as* $n\to\infty$.

L e m m a 7 ([6; 8], Lemma 3). *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma n^{-1}\lambda_n$ *is convergent, then*

$$\sum_{n=1}^{m} \log (n+1)\,\Delta\lambda_n = O(1),$$

as $m\to\infty$.

L e m m a 8. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma n^{-1}\lambda_n$ *is convergent, then*

$$\sum_{k=0}^{\left[\frac{n}{2}\right]-2} \left|\Delta\left\{\frac{(n+1)P_n-(k+1)P_k}{(n-k)} \log (n-k+1)\,\lambda_{n-k}\right\}\right|$$
$$= O\left[\log^2 (n+1)\,\lambda_{n-\left[\frac{n}{2}\right]+1}\right].$$

Proof. We observe that

$$\sum_{k=0}^{\left[\frac{n}{2}\right]-2} \left|\Delta\left\{\frac{(n+1)P_n-(k+1)P_k}{(n-k)} \log (n-k+1)\,\lambda_{n-k}\right\}\right|$$
$$= O(1) \sum_{k=0}^{\left[\frac{n}{2}\right]-2} \frac{P_{k+1}\log (n-k+1)\,\lambda_{n-k}}{(n-k)}$$
$$+ O(1) \sum_{k=0}^{\left[\frac{n}{2}\right]-2} \frac{\{(n+1)P_n-(k+1)P_k\}\log (n-k+1)\,\lambda_{n-k}}{(n-k)^2}$$
$$+ O(1) \sum_{k=0}^{\left[\frac{n}{2}\right]-2} \frac{(n+1)P_n-(k+1)P_k}{(n-k)} \log (n-k+1)\,(\lambda_{n-k-1}-\lambda_{n-k})$$

$$=O\left[\log^2(n+1)\lambda_{n-\left[\frac{n}{2}\right]+2}\sum_{k=0}^{\left[\frac{n}{2}\right]-2}\frac{1}{n-k}\right]$$

$$+O\left[\log^2(n+1)\lambda_{n-\left[\frac{n}{2}\right]+1}\right]$$

$$=O\left[\log^2(n+1)\lambda_{n-\left[\frac{n}{2}\right]+1}\right],$$

since $\{\lambda_n\}$ is non-negative and non-increasing and $\sum_{k=0}^{\left[\frac{n}{2}\right]-2}\frac{1}{n-k}=O(1)$.

4. *Proof of Theorem* 1. Since from (1.1)

$$t_n=\sum_{\nu=0}^{n}P_\nu\, u_{n-\nu}/P_n$$

where

$$u_n=\log(n+1)\lambda_n A_n(t)$$

we have

$$t_n-t_{n-1}=\sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n}-\frac{P_{\nu-1}}{P_{n-1}}\right)u_{n-\nu}$$

$$=\frac{1}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)u_{n-\nu}.$$

For the Fourier series of $f(t)$, at $t=x$,

$$A_n(x)=\frac{2}{\pi}\int_0^\pi\Phi(t)\cos nt\,dt$$

$$=\frac{2}{\pi}\Phi(\pi)\log\frac{r}{\pi}\int_0^\pi\frac{\cos nt}{\log\frac{r}{t}}\,dt$$

$$-\frac{2}{\pi}\int_0^\pi d\left\{\Phi(t)\log\frac{r}{t}\right\}\int_0^t\frac{\cos nu}{\log\frac{r}{u}}\,du$$

$$=-\frac{2}{n\pi}\Phi(\pi)\log\frac{r}{\pi}\int_0^\pi\left(\log\frac{r}{t}\right)^{-2}\frac{\sin nt}{t}\,dt$$

$$-\frac{2}{\pi}\int_0^\pi d\left\{\Phi(t)\log\frac{r}{t}\right\}\left\{\left(\log\frac{r}{u}\right)^{-1}\frac{\sin nu}{n}\right\}_0^t$$

$$+\frac{2}{n\pi}\int_0^\pi d\left\{\Phi(t)\log\frac{r}{t}\right\}\int_0^t\left(\log\frac{r}{u}\right)^{-2}\frac{\sin nu}{u}\,du,$$

and therefore

$$t_n - t_{n-1} = -\frac{2}{\pi}\,\Phi(\pi)\log\frac{r}{\pi}\,\frac{1}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)$$

$$\times\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\int_0^\pi\left(\log\frac{r}{t}\right)^{-2}\frac{\sin(n-\nu)\,t}{t}\,dt$$

$$-\frac{2}{\pi}\int_0^\pi d\left\{\Phi(t)\log\frac{r}{t}\right\}\left(\log\frac{r}{t}\right)^{-1}\frac{1}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)$$

$$\times\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\sin(n-\nu)\,t$$

$$+\frac{2}{\pi}\int_0^\pi d\left\{\Phi(t)\log\frac{r}{t}\right\}\frac{1}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)$$

$$\times\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\int_0^t\left(\log\frac{r}{u}\right)^{-2}\frac{\sin(n-\nu)\,u}{u}\,du.$$

Thus for establishing the theorem we have to establish that

(4.1)
$$\sum_{n=1}^{\infty}\frac{1}{P_n P_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\right.$$
$$\left.\times\int_0^\pi\left(\log\frac{r}{t}\right)^{-2}\frac{\sin(n-\nu)\,t}{t}\,dt\right|<\infty;$$

(4.2)
$$\sum_{n=1}^{\infty}\frac{1}{P_n P_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\right.$$
$$\left.\times\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\sin(n-\nu)\,t\right|=O\left[\log\frac{r}{t}\right];$$

(4.3)
$$\sum_{n=1}^{\infty}\frac{1}{P_n P_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\right.$$
$$\left.\times\int_0^t\left(\log\frac{r}{u}\right)^{-2}\frac{\sin(n-\nu)\,u}{u}\,du\right|<\infty,$$

since $\Phi(t)\log\frac{r}{t}$ is of bounded variation, $\Phi(\pi)\log\frac{r}{\pi}=O(1)$, and

$$\int_0^\pi\left|d\left\{\Phi(t)\log\frac{r}{t}\right\}\right|<\infty.$$

We write

$$\sum\nolimits_1 = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \frac{\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)} \right. \tag{4.4}$$

$$\left. \times \int_0^t \left(\log \frac{r}{u}\right)^{-2} \frac{\sin(n-\nu)u}{u} du \right|$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1/n+1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{\left[\frac{n}{2}\right]-1} \frac{(n+1)P_n - (\nu+1)P_\nu}{(n-\nu)(\nu+1)} \log(n-\nu+1)\lambda_{n-\nu} \right.$$

$$\left. \times \int_0^t \left(\log \frac{r}{u}\right)^{-2} \frac{\sin(n-\nu)u}{u} du \right|$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=\left[\frac{n}{2}\right]}^{n-1} \frac{\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)} \right.$$

$$\left. \times \int_0^t \left(\log \frac{r}{u}\right)^{-2} \frac{\sin(n-\nu)u}{u} du \right|$$

$$+ \sum_{n=1}^{\infty} \frac{1/n+1}{P_n P_{n-1}} \left| \sum_{\nu=\left[\frac{n}{2}\right]}^{n-1} \frac{(P_n - P_\nu)\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)} \right.$$

$$\left. \times \int_0^t \left(\log \frac{r}{u}\right)^{-2} \frac{\sin(n-\nu)u}{u} du \right|$$

$$= \Sigma_{1,1} + \Sigma_{1,2} + \Sigma_{1,3}, \text{ say.}$$

Now applying Abel's transformation to the inner sum in $\Sigma_{1,1}$ and making use of Lemmas 1 and 6 we have

$$\sum_{\nu=0}^{\left[\frac{n}{2}\right]-1} \frac{\{(n+1)P_n - (\nu+1)P_\nu\}\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)(\nu+1)} \tag{4.5}$$

$$\times \int_0^t \left(\log \frac{r}{u}\right)^{-2} \frac{\sin(n-\nu)u}{u} du$$

$$= O\left[\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{P_{\nu+1}\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)(\nu+1)} \sum_{\mu=0}^{\nu} \frac{1}{\{\log(n-\mu+1\}^2} \right]$$

$$+O\left[\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{\{(n+1)P_n-(\nu+1)P_\nu\}\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)^2(\nu+1)} \sum_{\mu=0}^{\upsilon} \frac{1}{\{\log(n-\mu+1)\}^2}\right]$$

$$+O\left[\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{\{(n+1)P_n-(\nu+1)P_\nu\}\log(n-\nu+1)(\lambda_{n-\nu-1}-\lambda_{n-\nu})}{(n-\nu)(\nu+1)}\right.$$

$$\left.\times\sum_{\mu=0}^{\nu} \frac{1}{\{\log(n-\mu+1)\}^2}\right]$$

$$+O\left[\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{\{(n+1)P_n-(\nu+1)P_\nu\}\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)(\nu+1)^2}\right.$$

$$\left.\times\sum_{\mu=0}^{\nu} \frac{1}{\{\log(n-\mu+1)\}^2}\right]+O\left[\lambda_{n-\left[\frac{n}{2}\right]+1}\right]$$

$$=O(1)\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{P_{\nu+1}\lambda_{n-\nu}}{(n-\nu)\{\log(n-\nu+1)\}}$$

$$+O(P_n)\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{\lambda_{n-\nu}}{(n-\nu)\log(n-\nu+1)}+O(P_n)\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{(\lambda_{n-\nu-1}-\lambda_{n-\nu})}{\log(n-\nu+1)}$$

$$+O(P_n)\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{\lambda_{n-\nu}}{(\nu+1)\log(n-\nu+1)}+O\left[\lambda_{n-\left[\frac{n}{2}\right]+1}\right]$$

$$=O\left[\lambda_{n-\left[\frac{n}{2}\right]+2}\right]\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \frac{1}{(n-\nu)}+O\left[\log(n+1)\,\lambda_{n-\left[\frac{n}{2}\right]+1}\right]$$

$$=O\left[\log(n+1)\,\lambda_{n-\left[\frac{n}{2}\right]+1}\right].$$

Now we consider

$$(4.6)\qquad \sum_{\nu=\left[\frac{n}{2}\right]}^{n-1} \frac{\log(n-\nu+1)\lambda_{n-\nu}}{(\upsilon+1)} \int_0^t \left(\log\frac{r}{u}\right)^{-2} \frac{\sin(n-\nu)u}{u}\,du$$

$$=O\left(\frac{1}{n}\right)\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{(n-\nu)}{\log(n-\nu+1)}\,\frac{\lambda_{n-\nu}}{(n-\nu)}$$

$$=O\left[\frac{1}{\log(n+1)}\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{\lambda_{n-\nu}}{n-\nu}\right]=O\left[\frac{1}{\log(n+1)}\right],$$

and also by Lemma 1,

$$\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{(P_n-P_\nu)\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\int_0^t\left(\log\frac{r}{u}\right)^{-2}\frac{\sin(n-\nu)u}{u}\,du \tag{4.7}$$

$$=O(1)\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{\lambda_{n-\nu}}{(n-\nu)\log(n-\nu+1)}=O(1).$$

Putting the estimates (4.5), (4.6) and (4.7) respectively in $\Sigma_{1,1}$, $\Sigma_{1,2}$ and $\Sigma_{1,3}$ we have

$$\sum\nolimits_{1,1}=O(1)\sum_{n=1}^{\infty}\frac{\lambda_{n-\left[\frac{n}{2}\right]+1}}{n\log(n+1)}=O(1),$$

$$\sum\nolimits_{1,2}=O(1)\sum_{n=1}^{\infty}\frac{1}{n\log^2(n+1)}=O(1),$$

and

$$\sum\nolimits_{1,3}=O(1)\sum_{n=1}^{\infty}\frac{1}{n\log^2(n+1)}=O(1),$$

which establishes that $\Sigma_1=O(1)$, and thus the truth of (4.3) is established.

Proof of (4.1) is similar to that of (4.3). Hence for establishing the theorem we have only to demonstrate the truth of (4.2).

Now let us consider

$$\sum\nolimits_2=\sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\sin(n-\nu)\,t\right| \tag{4.8}$$

$$\leqslant\sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{\left[\frac{n}{2}\right]-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\log(n-\nu+1)\,\lambda_{n-\nu}}{(n-\nu)}\sin(n-\nu)t\right|$$

$$+\sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{(P_n-P_\nu)}{(\nu+1)}\frac{\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)}\sin(n-\nu)t\right|$$

$$+\sum_{n=1}^{\infty}\frac{1/n+1}{P_nP_{n-1}}\left|\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{P_\nu\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)}\sin(n-\nu)t\right|$$

$$=\sum\nolimits_{2,1}+\sum\nolimits_{2,2}+\sum\nolimits_{2,3},\ \text{say.}$$

By Abel's transformation and Lemmas 5 and 8 we have

$$\sum\nolimits_{2,1}=\sum_{n=1}^{\infty}\frac{1/n+1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{\left[\frac{n}{2}\right]-1}\frac{\{(n+1)P_n-(\nu+1)P_\nu\}\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)}\frac{\sin(n-\nu)t}{(\nu+1)}\right| \tag{4.9}$$

$$=O\left[\sum_{n=1}^{\infty}\frac{1/n+1}{P_nP_{n-1}}\log\frac{r}{t}\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2}\left|\Delta\left[\frac{\{(n+1)P_n-(\nu+1)P_\nu\}\log(n-\nu+1)\lambda_{n-\nu}}{(n-\nu)}\right]\right|\right]$$

$$+O\left[\sum_{n=1}^{\infty}\frac{1/n+1}{P_nP_{n-1}}\log\frac{r}{t}\frac{\left\{(n+1)P_n-\left[\frac{n}{2}\right]P_{\left[\frac{n}{2}\right]-1}\right\}\log\left(n-\left[\frac{n}{2}\right]+2\right)\lambda_{n-\left[\frac{n}{2}\right]+1}}{\left(n-\left[\frac{n}{2}\right]+1\right)}\right]$$

$$=O\left[\log\frac{r}{t}\sum_{n=1}^{\infty}\frac{1/n+1}{P_nP_{n-1}}\log^2(n+1)\lambda_{n-\left[\frac{n}{2}\right]+1}\right]$$

$$=O\left(\log\frac{r}{t}\right)\sum_{n=1}^{\infty}\frac{\lambda_{n-\left[\frac{n}{2}\right]+1}}{(n+1)}=O\left[\log\frac{r}{t}\right].$$

By making use of Lemmas 3 and 7 we have

$$\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{(P_n-P_\nu)\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)}\frac{\sin(n-\nu)t}{(n-\nu)} \tag{4.10}$$

$$=\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\Delta\left[\frac{(P_n-P_\nu)\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)}\right]\sum_{\mu=0}^{\nu}\frac{\sin(n-\mu)t}{(n-\mu)}$$

$$\frac{\left(P_n-P_{\left[\frac{n}{2}\right]}\right)\log\left(n-\left[\frac{n}{2}\right]+1\right)\lambda_{n-\left[\frac{n}{2}\right]}}{\left(\left[\frac{n}{2}\right]+1\right)}\sum_{\mu=0}^{\left[\frac{n}{2}\right]-1}\frac{\sin(n-\mu)t}{(n-\mu)}$$

$$+\frac{(P_n-P_{n-1})\cdot\log 2\cdot\lambda_1}{n}\sum_{\mu=0}^{n-1}\frac{\sin(n-\mu)t}{(n-\mu)}$$

$$=O\left[\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\frac{\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)^2}\right]+O\left[\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\frac{(P_n-P_\nu)\lambda_{n-\nu}}{(\nu+1)(n-\nu)}\right]$$

$$+O\left[\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\frac{(P_n-P_\nu)\log(n-\nu+1)(\lambda_{n-\nu-1}-\lambda_{n-\nu})}{(\nu+1)}\right]+O\left(\frac{1}{n}\right)$$

$$=O(1)\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\frac{1}{(\nu+1)^2}+O\left(\frac{1}{n}\right)\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\frac{\lambda_{n-\nu}}{(n-\nu)}+O\left(\frac{1}{n}\right)\left|\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\log(n-\nu+1)\Delta\lambda_{n-\nu}\right|$$

$$+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right),$$

and therefore

$$\sum\nolimits_{2,2}=O(1)\sum_{n=1}^{\infty}\frac{1}{n\log^2(n+1)}=O(1). \tag{4.11}$$

Since $|\sin(n-\nu)t|\leqslant nt$

$$\left|\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{P_\nu\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)}\sin(n-\nu)t\right|\leqslant Knt\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{P_\nu}{(\nu+1)}=O(nP_n t), \tag{4.12}$$

Again since $\Sigma\sin nt=O(t^{-1})$, we have

$$\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{P_\nu\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)}\sin(n-\nu)t \tag{4.13}$$

$$=\sum_{\nu=\left[\frac{n}{2}\right]}^{n-2}\Delta\left\{\frac{P_\nu\log(n-\nu+1)\lambda_{n-\nu}}{(\nu+1)}\right\}\sum_{\mu=0}^{\nu}\sin(n-\mu)t$$

$$-\frac{P_{\left[\frac{n}{2}\right]}\log\left(n-\left[\frac{n}{2}\right]+1\right)\lambda_{n-\left[\frac{n}{2}\right]}}{\left(\left[\frac{n}{2}\right]+1\right)}\sum_{\mu=0}^{\left[\frac{n}{2}\right]-1}\sin(n-\mu)t$$

$$+\frac{P_{n-1}\cdot\log 2\cdot\lambda_1}{n}\sum_{\mu=0}^{n-1}\sin(n-\mu)t$$

$$= O\left[\frac{\log (n+1)}{t}\right] \sum_{\nu=\left[\frac{n}{2}\right]}^{n-2} \frac{\log (n-\nu+1) \lambda_{n-\nu}}{(\nu+1)^2}$$

$$+ O\left(\frac{1}{t}\right) \sum_{\nu=\left[\frac{n}{2}\right]}^{n-2} \frac{P_\nu \lambda_{n-\nu}}{(\nu+1)(n-\nu)}$$

$$+ O\left(\frac{1}{t}\right) \left| \sum_{\nu=\left[\frac{n}{2}\right]}^{n-2} \frac{P_\nu \log (n-\nu+1) \Delta \lambda_{n-\nu}}{(\nu+1)} \right| + O\left[\frac{\log (n+1)}{nt}\right]$$

$$= O\left[\frac{\log (n+1)}{nt}\right] \sum_{\nu=\left[\frac{n}{2}\right]}^{n-2} \frac{1}{(\nu+1)} + O\left[\frac{\log (n+1)}{nt}\right] \sum_{\nu=\left[\frac{n}{2}\right]}^{n-2} \frac{\lambda_{n-\nu}}{n-\nu}$$

$$+ O\left[\frac{\log (n+1)}{nt}\right] \left| \sum_{\nu=\left[\frac{n}{2}\right]}^{n-2} \log (n-\nu+1) \Delta \lambda_{n-\nu} \right| + O\left[\frac{\log (n+1)}{nt}\right]$$

$$= O\left[\frac{\log (n+1)}{nt}\right],$$

by an application of Lemmas 6 and 7.

Taking $\tau = [t^{-1}]$, let us write $\Sigma_{2,3}$ in the form

$$\Sigma_{2,3} = \sum_{n=1}^{\tau} + \sum_{n=\tau+1}^{\infty} = \Sigma_{2,3,1} + \Sigma_{2,3,2}, \text{ say.} \tag{4.14}$$

Then by (4.12)

$$\Sigma_{2,3,1} = O(1) \sum_{n=1}^{\tau} \frac{1/n+1}{P_n P_{n-1}} n P_n t = O(t\tau) = O(1), \tag{4.15}$$

and from (4.13) we have

$$\Sigma_{2,3,2} = O\left(\frac{1}{t}\right) \sum_{n=\tau+1}^{\infty} \frac{1}{n^2 \log (n+1)} = O\left[\frac{1}{t\tau \log \tau}\right] = O(1). \tag{4.16}$$

Combining the estimates (4.8), (4.9), (4.11) (4.14), (4.15) and (4.16) we have

$$\Sigma_2 = O\left[\log \frac{r}{t}\right],$$

which proves (4.2)

This completes the proof of Theorem 1.

5. *Proof of Theorem* 2. We have

$$A_n(x)=\frac{2}{\pi}\int_0^{\pi}\Phi(t)\cos nt\,dt$$

$$=\frac{2}{\pi}\int_0^{\pi}\Phi(t)\left(\log\frac{r}{t}\right)^{1-\delta}\frac{\cos nt}{\left(\log\frac{r}{t}\right)^{1-\delta}}\,dt$$

$$=-\frac{2(1-\delta)}{n\pi}\Phi(\pi)\left(\log\frac{r}{\pi}\right)^{1-\delta}\int_0^{\pi}\left(\log\frac{r}{t}\right)^{-2+\delta}\frac{\sin nt}{t}\,dt$$

$$-\frac{2}{\pi}\int_0^{\pi}d\left\{\Phi(t)\left(\log\frac{r}{t}\right)^{1-\delta}\right\}\left(\log\frac{r}{t}\right)^{-1+\delta}\frac{\sin nt}{n}$$

$$+\frac{2(1-\delta)}{n\pi}\int_0^{\pi}d\left\{\Phi(t)\left(\log\frac{r}{t}\right)^{1-\delta}\right\}\int_0^{t}\left(\log\frac{r}{u}\right)^{-2+\delta}\frac{\sin nu}{u}\,du$$

Proceeding as in the proof of Theorem 1, for establishing this theorem we have to establish that

$$(5.1)\qquad \sum_{n=1}^{\infty}\frac{1}{P_n P_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{1}{(n-\nu)\{\log(n-\nu+1)\}^{\delta}}\right.$$
$$\left.\times\int_0^{\pi}\left(\log\frac{r}{u}\right)^{-2+\delta}\frac{\sin(n-\nu)u}{u}\,du\right|<\infty\ (0<\delta<1);$$

$$(5.2)\qquad \sum_{n=1}^{\infty}\frac{1}{P_n P_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^{\delta}}\right|$$
$$=O\left[\left(\log\frac{r}{t}\right)^{1-\delta}\right],\ (0<\delta\leqslant 1);$$

$$(5.3)\qquad \sum_{n=1}^{\infty}\frac{1}{P_n P_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{1}{(n-\nu)\{\log(n-\nu+1)\}^{\delta}}\right.$$
$$\left.\times\int_0^{t}\left(\log\frac{r}{u}\right)^{-2+\delta}\frac{\sin(n-\nu)u}{u}\,du\right|<\infty\ (0<\delta<1).$$

The proof of (5.1) and (5.3) follows in quite a similar fashion as those of (4.1) and (4.3) by taking $\lambda_n=1/\{\log(n+1)\}^{\delta+1}$ and using Lemma 2 in place of Lemma 1. The proof of (5.2) is a bit distinct from that of (4.2) and so we give it here.

Let us write

$$(5.4)\qquad \sum{}' = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \right|$$

$$= \left(\sum_{n=1}^{\tau} + \sum_{n=\tau+1}^{\infty} \right) \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \right|$$

$$= \sum{}_1' + \sum{}_2', \text{ say.}$$

Now

$$(5.5)\qquad \sum{}_1' = \sum_{n=1}^{\tau} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \right|$$

$$\leqslant Kt \sum_{1}^{\tau} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_n}{\nu+1} = Kt\,\tau = O(1).$$

Again let

$$(5.6)\qquad \sum{}_2' = \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \right|$$

$$\leqslant \sum_{n=\tau+1}^{\infty} \frac{1/n+1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{\left[\frac{n}{2}\right]-1} \frac{(n+1)P_n - (\nu+1)P_\nu}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \frac{\sin(n-\nu)t}{(\nu+1)} \right|$$

$$+ \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=\left[\frac{n}{2}\right]}^{n-1} \frac{P_n - P_\nu}{(\nu+1)} \frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \right|$$

$$+ \sum_{n=\tau+1}^{\infty} \frac{1/n+1}{P_n P_{n-1}} \left| \sum_{\nu=\left[\frac{n}{2}\right]}^{n-1} \frac{P_\nu}{(\nu+1)} \frac{\sin(n-\nu)t}{\{\log(n-\nu+1)\}^\delta} \right|$$

$$= \sum{}_{2,1}' + \sum{}_{2,2}' + \sum{}_{2,3}', \text{ say.}$$

Then

$$(5.7)\qquad \sum{}_{2,1}' = \sum_{n=\tau+1}^{\infty} \frac{1/n+1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{\left[\frac{n}{2}\right]-1} \frac{(n+1)P_n - (\nu+1)P_\nu}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \frac{\sin(n-\nu)t}{(\nu+1)} \right|$$

$$= O\left[\sum_{n=\tau+1}^{\infty} \frac{1/n+1}{P_n P_{n-1}} \log\frac{r}{t} \sum_{\nu=0}^{\left[\frac{n}{2}\right]-2} \left| \Delta\left\{ \frac{(n+1)P_n - (\nu+1)P_\nu}{(n-\nu)\{\log(n-\nu+1)\}^\delta} \right\} \right| \right]$$

$$+O\left[\sum_{n=\tau+1}^{\infty}\frac{1/n+1}{P_n P_{n-1}}\log\frac{r}{t}\,\frac{(n+1)P_n-\left(\left[\frac{n}{2}\right]\right)P_{\left[\frac{n}{2}\right]-1}}{\left(n-\left[\frac{n}{2}\right]+1\right)\left\{\log\left(n-\left[\frac{n}{2}\right]+2\right)\right\}^{\delta}}\right]$$

$$=O\left[\log\frac{r}{t}\right]\sum_{n=\tau+1}^{\infty}\frac{1/n+1}{P_n P_{n-1}}\{\log(n+1)\}^{1-\delta}=O\left[\left(\log\frac{r}{t}\right)^{1-\delta}\right],$$

by Lemma 5 and the estimate

$$\sum_{\nu=0}^{\left[\frac{n}{2}\right]-2}\left|\Delta\left\{\frac{(n+1)P_n-(\nu+1)P_\nu}{(n-\nu)\{\log(n-\nu+1)\}^{\delta}}\right\}\right|=O\left[\{\log(n+1)\}^{1-\delta}\right]\ (0<\delta\leqq 1);$$

which is easily obtained by putting $\lambda_n=1/\{\log(n+1)\}^{1+\delta}$ in Lemma 8.

Again putting $\lambda_n=1/\{\log(n+1)\}^{\delta+1}$ in (4.10) and (4.13) we get

$$\left|\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{(P_n-P_\nu)}{(\nu+1)}\,\frac{\sin(n-\nu)t}{(n-\nu)\{\log(n-\nu+1)\}^{\delta}}\right|=O\left(\frac{1}{n}\right),$$

and

$$\left|\sum_{\nu=\left[\frac{n}{2}\right]}^{n-1}\frac{P_\nu}{(\nu+1)}\,\frac{\sin(n-\nu)t}{\{\log(n-\nu+1)\}^{\delta}}\right|=O\left[\frac{P_n}{nt}\right],$$

and therefore

(5.8) $$\sum\nolimits_{2,2}'=O(1)\sum_{n=\tau+1}^{\infty}\frac{1}{n\log^2(n+1)}=O(1),$$

(5.9) $$\sum\nolimits_{2,3}'=O\left(\frac{1}{t}\right)\sum_{n=\tau+1}^{\infty}\frac{1}{n^2\log(n+1)}=O\left(\frac{1}{t\,\tau\log\tau}\right)=O(1).$$

Collecting our results in (5.4), (5.5), (5.6), (5.7), (5.8) and (5.9) we have

$$\Sigma'=O\left[\left(\log\frac{r}{t}\right)^{1-\delta}\right]\ (0<\delta\leqq 1),$$

which establishes (5.2). Hence the theorem.

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