

ON EULERIAN INTEGRALS ASSOCIATED WITH KAMPÉ DE FÉRIET'S FUNCTION

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1. Introductory

In the present note we evaluate three Eulerian integrals of the first kind that involve a generalized Kampé de Fériet function in two arguments. The first one is intended to unify the two main integral formulas (1) and (1a) proved recently in the *Mathematische Zeitschrift* (cf. [13], pp. 119, 121) which, in turn, reduce to the earlier results of Buschman [4], while the others provide us with elegant extensions of scores of hitherto scattered results in the theory of generalized hypergeometric functions.

2. The definite integrals

Making use of the familiar abbreviation introduced by one of us in [8] and [9], let (a) denote the sequence of A parameters a_1, a_2, \dots, a_A ; i.e. unless otherwise stated, it is understood that there are A of the a parameters, B of the b parameters, B' of the b' parameters, and so on.

Also let

$$(2.1) \quad S_{C:D; D'}^{A:B; B'} \left(\begin{matrix} x \\ y \end{matrix} \right) = S_{C:D; D'}^{A:B; B'} \left([(a): \theta, \Phi]; [(b): \psi]; [b'): \psi']; x, y \right) \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^A \Gamma[a_j + m\theta_j + n\Phi_j] \prod_{j=1}^B \Gamma[b_j + m\psi_j] \prod_{j=1}^{B'} \Gamma[b'_j + n\psi'_j]}{\prod_{j=1}^C \Gamma[c_j + m\delta_j + n\varepsilon_j] \prod_{j=1}^D \Gamma[d_j + m\eta_j] \prod_{j=1}^{D'} \Gamma[d'_j + n\eta'_j]} \frac{x^m}{m!} \frac{y^n}{n!},$$

where, for convergence,

$$\begin{cases} 1 + \sum_{j=1}^C \delta_j + \sum_{j=1}^D \eta_j - \sum_{j=1}^A \theta_j - \sum_{j=1}^B \psi_j > 0, \\ 1 + \sum_{j=1}^C \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^A \Phi_j - \sum_{j=1}^{B'} \psi'_j > 0; \end{cases}$$

so that when $y \rightarrow 0$, $S\left(\begin{matrix} x \\ y \end{matrix}\right)$ reduces to the generalized hypergeometric series introduced by Wright (cf. [14] and [15]), and when the positive real constants $\theta_1, \dots, \theta_A; \Phi_1, \dots, \Phi_A; \psi_1, \dots, \psi_B; \psi'_1, \dots, \psi'_{B'}; \delta's; \varepsilon's; \eta's; \eta''s$ are all taken as unity it equals

$$\frac{\prod_{j=1}^A \Gamma[a_j] \prod_{j=1}^B \Gamma[b_j] \prod_{j=1}^{B'} \Gamma[b'_j]}{\prod_{j=1}^C \Gamma[c_j] \prod_{j=1}^D \Gamma[d_j] \prod_{j=1}^{D'} \Gamma[d'_j]} F\left[\begin{matrix} (a):(b); (b'); & x, y \\ (c):(d); (d'); & \end{matrix}\right],$$

where $F[x, y]$ denotes the Kampé de Fériet function [1, p. 150] in the contracted notation of Burchnall and Chaundy [3, p. 112].

Then, as an immediate consequence of the well-known formula [6, Vol. I, p. 311 (31)]

$$(2.2) \quad \int_0^1 t^{\alpha-1} (1-t^\lambda)^{\beta-1} dt = \frac{\Gamma\left[\frac{\alpha}{\lambda}\right] \Gamma[\beta]}{\lambda \Gamma\left[\frac{\alpha}{\lambda} + \beta\right]}, \quad \lambda > 0, \quad \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0,$$

we obtain our first result in the form

$$(2.3) \quad \begin{aligned} & \int_0^1 t^{\alpha-1} (1-t^\lambda)^{\beta-1} S\left[\begin{matrix} A:B; B' \\ C:D; D' \end{matrix}\right] \left(\begin{matrix} xt^\mu[1-t^\lambda]^\rho \\ yt^\nu[1-t^\lambda]^\sigma \end{matrix} \right) dt \\ &= \frac{1}{\lambda} S\left[\begin{matrix} A+2:B; B' \\ C+1:D; D' \end{matrix}\right] \left(\begin{matrix} [(a):\theta, \Phi], [\alpha\lambda^{-1}:\mu\lambda^{-1}, \nu\lambda^{-1}], [\beta:\rho, \sigma]: \\ [(c):\delta, \varepsilon], [\alpha\lambda^{-1}+\beta:\mu\lambda^{-1}+\rho, \nu\lambda^{-1}+\sigma]: \\ [(b):\psi]; [(b'):\psi']; x, y \\ [(d):\eta]; [(d'):\eta']; \end{matrix} \right), \end{aligned}$$

provided $\lambda > 0$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and the series on the right converges.

Next we make use of the definition (2.1) together with a generalization of (2.2), viz.

$$(2.4) \quad \begin{aligned} & \int_0^1 t^{\sigma-1} (1-t^\lambda)^{\beta-1} {}_2F_1[-n, \alpha+\beta+n-1; \alpha; t^\lambda] dt \\ &= \frac{1}{\lambda} \frac{\Gamma\left[\frac{\alpha}{\lambda}\right] \Gamma\left[\alpha - \frac{\sigma}{\lambda} + n\right] \Gamma[\alpha] \Gamma[\beta+n]}{\Gamma[\alpha+n] \Gamma\left[\alpha - \frac{\sigma}{\lambda}\right] \Gamma\left[\beta + \frac{\sigma}{\lambda} + n\right]}, \end{aligned}$$

$$\lambda > 0, \quad \operatorname{Re}(\sigma) > 0, \quad \operatorname{Re}(\beta) > 0, \quad n = 0, 1, 2, \dots,$$

which follows readily from the known integral (2), p. 284 in [6, Vol. II], and we find that

$$(2.5) \quad \int_0^1 t^{\sigma-1} (1-t^\lambda)^{\beta-1} {}_2F_1[-n, \alpha+\beta+n-1; \alpha; t^\lambda] \cdot S \begin{matrix} A:B; & B' \\ C:D; & D' \end{matrix} \left(\frac{xt^\mu}{yt^\nu} \right) dt$$

$$= \frac{(-)^n \Gamma[\alpha] \Gamma[\beta+n]}{\lambda \Gamma[\alpha+n]} S \begin{matrix} A+2:B; & B' \\ C+2:D; & D' \end{matrix} \left(\begin{matrix} [(a):\theta, \Phi], [\sigma\lambda^{-1}:\mu\lambda^{-1}, \nu\lambda^{-1}], \\ [(c):\delta, \varepsilon], [\beta+\sigma\lambda^{-1}+n:\mu\lambda^{-1}, \nu\lambda^{-1}], \\ [1-\alpha+\sigma\lambda^{-1}-n:\mu\lambda^{-1}, \nu\lambda^{-1}]:[(b):\psi]; [(b'):\psi']; \\ [1-\alpha+\sigma\lambda^{-1}:\mu\lambda^{-1}, \nu\lambda^{-1}]:[(d):\eta]; [(d'):\eta'] \end{matrix}; \begin{matrix} x, y \end{matrix} \right),$$

valid within the domain of convergence of the resulting series when $\lambda>0$, $\operatorname{Re}(\sigma)>0$, $\operatorname{Re}(\beta)>0$, and n is a non-negative integer.

The third result, which involves the associated Legendre function of the first kind, follows similarly from Barnes's integral [5, p. 172 (24)]

$$(2.6) \quad \int_0^1 t^{\lambda-1} (1-t^2)^{-\frac{1}{2}\mu} P_v^\mu(t) dt$$

$$= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma[\lambda]}{\Gamma\left[\frac{1}{2}(\lambda-\mu-\nu+1)\right] \Gamma\left[\frac{1}{2}(\lambda-\mu+\nu+2)\right]},$$

$\operatorname{Re}(\lambda)>0$, $\operatorname{Re}(\mu)<1$, and we have

$$(2.7) \quad \int_0^1 t^{\lambda-1} (1-t^2)^{-\frac{1}{2}\mu} P_v^\mu(t) S \begin{matrix} A:B; & B' \\ C:D; & D' \end{matrix} \left(\frac{xt^{2\rho}}{yt^{2\sigma}} \right) dt$$

$$= 2^{\mu-1} S \begin{matrix} A+2:B; & B' \\ C+2:D; & D' \end{matrix} \left(\begin{matrix} [(a):\theta, \Phi], \left[\frac{1}{2}\lambda:\rho, \sigma \right], \left[\frac{1}{2}\lambda+\frac{1}{2}:\rho, \sigma \right]: \\ [(c):\delta, \varepsilon], \left[\frac{1}{2}(\lambda-\mu-\nu+1):\rho, \sigma \right], \left[\frac{1}{2}(\lambda-\mu+\nu+2):\rho, \sigma \right]: \\ [(b):\psi]; [(b'):\psi']; \\ [(d):\eta]; [(d'):\eta'] \end{matrix}; \begin{matrix} x, y \end{matrix} \right),$$

provided the double series on the right-hand side converges, $\operatorname{Re}(\lambda)>0$, and $\operatorname{Re}(\mu)<1$.

3. Particular cases

When $n\rightarrow 0$, (2.5) evidently reduces to the special case $\rho=\sigma=0$ of (2.3). Moreover, if in (2.3) we set the positive real constants

$$\theta_1, \dots, \theta_A; \Phi_1, \dots, \Phi_A; \psi_1, \dots, \psi_B; \psi'_1, \dots, \psi'_{B'}; \delta_1, \dots, \delta_C;$$

$$\varepsilon_1, \dots, \varepsilon_C; \eta_1, \dots, \eta_D; \eta'_1, \dots, \eta'_{D'}$$

equal to unity each, in addition to letting either

- (i) $\rho = \sigma = 0, \mu = \nu = n\lambda$, or
- (ii) $\nu = \rho = 0, \mu\lambda^{-1} = \sigma = n$,

we shall readily obtain the main formulas (1) and (1a) of [13] which, in turn, specialize into scores of hitherto scattered results (see, for instance, [13], §§ 4, 5, and 6; and [6], Vol. II, pp. 398—400(1) to (12)) including those of Buschman [4] proved earlier in the same Zeitschrift.

It may be of interest to conclude with the remark that several particular forms of our formulas (2.5) and (2.7) occur also in [2], [7], [10], [11], and [12].

REFERENCES

- [1] P. Appell et J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Gauthier-Villars, Paris, 1926.
- [2] B. R. Bhonsle, *On some results involving Jacobi polynomials*, J. Indian Math. Soc. (New Ser.), 26 (1962), 187—190.
- [3] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions-II*, Quart. J. Math. (Oxford Ser.), 12 (1941), 112—128.
- [4] R. G. Buschman, *Integrals of hypergeometric functions*, Math. Zeitschr. 89 (1965), 74 — 76.
- [5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, 1954.
- [6] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of integral transforms*, Vols. I & II, McGraw-Hill, New York, 1954.
- [7] H. M. Srivastava, *Some integrals involving products of Bessel and Legendre functions*, Rend. Sem. Mat. Univ. Padova 35 (1965), 418—423.
- [8] H. M. Srivastava, *The integration of generalized hypergeometric functions*, Proc. Cambridge Philos. Soc. 62 (1966), 761—764.
- [9] H. M. Srivastava, *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos. Soc. 63 (1967), 425—429.
- [10] H. M. Srivastava, *Some integrals involving products of Bessel and Legendre functions-II*, Rend. Sem. Mat. Univ. Padova 37 (1967), 1—10.
- [11] H. M. Srivastava, *Integration of certain products containing Jacobi polynomials*, Collectanea Math. 19 (1968), 3—9.
- [12] H. M. Srivastava, *A note on a generalization of Sonine's first finite integral* Matematiche, 23 (1968), 1—6.
- [13] G. P. Srivastava and S. Saran, *Integrals involving Kampé de Fériet function*, Math. Zeitschr. 98 (1967), 119—125.
- [14] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc. 10 (1935), 286—293.
- [15] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. (2), 46 (1940), 389—408.

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