PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 9 (23), 1969. pp. 189—198.

ANALYTIC AND C-ANALYTIC FUNCTIONS

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(Communicated November 22, 1968)

I

1. For a complex function

$$w=u(x, y)+i\,v(x, y)$$

where u, v are differentiable real functions, the operator B can be defined: [1]

$$B \stackrel{\text{def}}{=} \frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}.$$

In other words

$$Bw=\frac{\partial w}{\partial x}+i\,\frac{\partial w}{\partial y}=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

The following properties of the operator B can be easily checked:

$$B(w_1+w_2)=Bw_1+Bw_2$$

$$Bw_1 w_2=w_1 Bw_2+w_2 Bw_1$$

$$Bz=0$$

$$B\bar{z}=2$$

$$Bf(w)=f'(w)\,Bw$$

$$Bf(w_1, w_2)=\frac{\partial f}{\partial w_1}\,Bw_1+\frac{\partial f}{\partial w_2}\,Bw_2$$

As a special case, we have

$$(1)\qquad Bw(z, \bar{z})=\frac{\partial w}{\partial z}\,Bz+\frac{\partial w}{\partial \bar{z}}\,B\bar{z}=2\,\frac{\partial w}{\partial \bar{z}}.$$

Besides the operator B, an other operator, operator C, can be defined for the function w

$$C \stackrel{\text{def}}{=} \frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y};$$

in other words

$$Cw = \frac{\partial w}{\partial x} - i\,\frac{\partial w}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Operator C has properties analogous to those of B:

$$C(w_1 + w_2) = Cw_1 + Cw_2$$

$$Cw_1 w_2 = w_1 Cw_2 + w_2 Cw_1$$

$$Cz = 2$$

$$C\bar{z} = 0$$

$$Cf(w) = f'(w)\, Cw$$

$$Cf(w_1, w_2) = \frac{\partial f}{\partial w_1}\, Cw_1 + \frac{\partial f}{\partial w_2}\, Cw_2.$$

As a special case, we have

$$Cw(z, \bar{z}) = \frac{\partial w}{\partial z}\, Cz + \frac{\partial w}{\partial \bar{z}}\, C\bar{z} = 2\,\frac{\partial w}{\partial z}. \tag{1'}$$

2. In his doctorate dissertation B. Riemann quotes the following formula [2]

$$\frac{dw}{dz} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] + \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] e^{-2i\varphi} \tag{2}$$

where $dz = \varepsilon\, e^{i\varphi}$.

If the function w is thought of as a function of two *independent* variables z and $\bar{z}$, i. e. $w = w(z, \bar{z})$, by (1) and (1') we see that formula (2) has the following form

$$\frac{dw}{dz} = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}}\, e^{-2i\varphi}$$

i. e.
$$dw = \frac{\partial w}{\partial z}\, dz + \frac{\partial w}{\partial \bar{z}}\, \varepsilon\, e^{-i\varphi}, \text{ or [3]}$$

$$dw = \frac{\partial w}{\partial z}\, dz + \frac{\partial w}{\partial \bar{z}}\, d\bar{z} \tag{3}$$

The expression $\dfrac{dw}{dz}$ will be independent of the direction φ if and only if $\dfrac{\partial w}{\partial \bar{z}} = 0$.

This condition represents the famous Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0; \qquad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0. \tag{4}$$

A complex function w whose real and imaginary parts satisfy equations (4) is called, as usual, an analytic function. For such a function we shall say that it belongs to class A, i. e.

$$w \in A \overset{\text{def}}{\Leftrightarrow} \frac{\partial w}{\partial \bar{z}}=0.$$

Consider again formula (3). Dividing by $d\bar{z}$ we get

$$\frac{dw}{d\bar{z}}=\frac{\partial w}{\partial z}\frac{dz}{d\bar{z}}+\frac{\partial w}{\partial \bar{z}}$$

i. e.

$$\frac{dw}{d\bar{z}}=\frac{\partial w}{\partial z}e^{2i\varphi}+\frac{\partial w}{\partial \bar{z}}.$$

Clearly, the expression $\frac{dw}{d\bar{z}}$ will be independent of the direction φ if and only if $\frac{\partial w}{\partial z}=0$, or

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0; \qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{5}$$

For the function w which has this property we shall say that it is conjugately analytic (c-analytic in further text), or that it belongs to class $\bar{A}$; in other words

$$w \in \bar{A} \overset{\text{def}}{\Leftrightarrow} \frac{\partial w}{\partial z}=0.$$

Theorem 1. *The only elements of the set* $A \cap \bar{A}$ *are constants.*

This can easily be seen by solving the system (4) — (5).

Theorem 2. *A necessary and sufficient condition for the function* $w(z,\bar{z})$ *to be analytic is that it does not depend on* $\bar{z}$.

Proof. $w(z,\bar{z}) \in A$ *if and only if* $\frac{\partial w}{\partial \bar{z}}=0$, *which means that* w *does not depend on* $\bar{z}$.

Theorem 2'. *A necessary and sufficient condition for the function* $w(z,\bar{z})$ *to be c-analytic is that it does not depend on* z.

In general case, $w(z,\bar{z}) \neq \overline{w(\bar{z},z)}$, even if $w \in A$.

(In fact if $w(z,\bar{z})=u(x,y)+iv(x,y)$, then

$$w(z,\bar{z})=\overline{w(z,\bar{\bar{z}})}$$

if and only if $u(x,y)$ is even and $v(x,y)$ is odd, both with respect to y.) However, if $f(z)\in A$, then both $f(\bar{z})$ and $\overline{f(z)}$ belong to $\bar{A}$ (even though they are not equal).

Theorem 3. *Let $f(z)\in A$. Then $f(\bar{z})\in \bar{A}$.*

Proof. *By assumption, $f(z)\in A$ and therefore $f(z)$ does not depend on $\bar{z}$ (theorem 2). Replacing each appearance of the variable z by $\bar{z}$, we obtain the function $f(\bar{z})$ which does not depend on z, and according to theorem 2′, $f(\bar{z})\in\bar{A}$.*

Theorem 4. *Let $f(z)\in A$. Then $\overline{f(z)}\in\bar{A}$.*

Proof. Let $f(z)=u(x,y)+iv(x,y)$. Then $\overline{f(z)}=u(x,y)-iv(x,y)$. Since $f(z)\in A$, we have

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0; \qquad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0.$$

Denote $-v(x,y)$ by $V(x,y)$. Then $\overline{f(z)}=u(x,y)+iV(x,y)$, and for the functions u, V we have

$$\frac{\partial u}{\partial x}+\frac{\partial V}{\partial y}=0; \qquad \frac{\partial V}{\partial x}-\frac{\partial u}{\partial y}=0,$$

which means that $\overline{f(z)}\in\bar{A}$.

The following analogues can also be easily proved:

Theorem 3′. *Let $f(\bar{z})\in\bar{A}$. Then $f(z)\in A$.*

Theorem 4′. *Let $f(\bar{z})\in\bar{A}$. Then $\overline{f(\bar{z})}\in A$.*

According to the definitions of operators B, C, we have:

$$w\in A \Leftrightarrow Bw=0$$

$$w\in\bar{A}\Leftrightarrow Cw=0.$$

Besides these operators, we can introduce the operators $\bar{B}$, $\bar{C}$:

$$\bar{B} \overset{\text{def}}{=} \frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}-i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),$$

$$\bar{C} \overset{\text{def}}{=} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-i\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right);$$

Therefore, a function w is analytic if and only if one of the following conditions holds:

$$Bw=0, \quad \bar{B}w=0, \quad C\bar{w}=0, \quad \bar{C}\bar{w}=0.$$

Analogously, a function w is c-analytic if and only if one of the following conditions holds:

$$Cw=0, \quad \bar{C}w=0, \quad B\bar{w}=0, \quad \bar{B}\bar{w}=0.$$

3. We state two more obvious theorems.

Theorem 5. *If $w \in \bar{A}$, then there is one and only one function $f(z) \in A$ such that $w = f(\bar{z})$.*

Theorem 5'. *If $w \in A$, then there is one and only one function $f(\bar{z}) \in \bar{A}$ such that $w = f(z)$.*

According to the above theorems, it is clear that there is a certain "isomorphism" between the sets A and $\bar{A}$, realized by a formal replacement of the symbol z by the symbol $\bar{z}$. Having this in mind it is not difficult to see that all the theorems of the theory of analytic functions have their analogue in the theory of c-analytic functions. Also, all the concepts defined in the theory of analytic functions can be introduced for c-analytic functions, e. g. integral, isolated singularity, residue, etc. It is convenient to call them c-integral, c-isolated singularity, c-residue, etc.

As an example, we quote the Cauchy-Goursat theorem for c-analytic functions.

Theorem 6. *If a function $f(\bar{z})$ is c-analytic in a simply-connected region R, and if C is a closed contour lying entirely within R, then*

$$\int_C f(\bar{z})\, d\bar{z} = 0.$$

Proof. Let $\bar{f}(z) = u(x, y) + i\, v(x, y)$. Then

$$\int_C f(\bar{z})\, d\bar{z} = \int_C (u + i\, v)\;(dx - idy) = \int_C udx + vdy + i \int_C vdx - vdy.$$

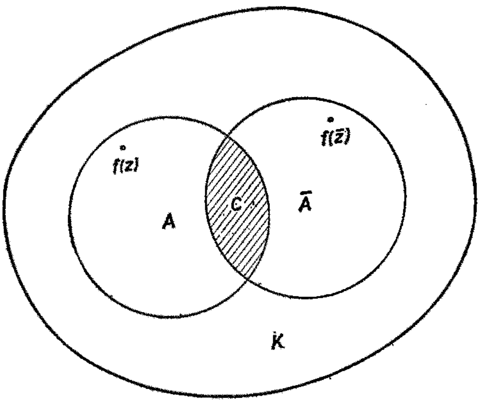
Using Green's theorem, we have

$$\int_C f(\bar{z})\, d\bar{z} = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy - i \iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy$$

the double integrals being taken over the area enclosed by C.

Considering equations (5) we conclude that $\int_C f(\bar{z})\, d\bar{z} = 0$.

The following schema illustrates the relations between the sets $K, A, A, \bar{C}$, where K denotes the set of all complex functions and C the set of all constants. To every function $f(z) \in A$, corresponds one and only one function $f(\bar{z}) \in \bar{A}$, and vice versa. Constants remain fixed under that "isomorphism".



II

1. It has been shown [4] that Goursat's functions

$$G(z, \bar{z}) = f(z) + \bar{z}\, g(z); \quad f,\ g \in A$$

have the property that their real and imaginary parts satisfy Maxwell's equations

$$\Delta^2 u = 0, \quad \Delta^2 v = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It has also been shown that they are the only non-analytic functions whose *deviation from being analytic*, B, is an analytic function, i. e. that they are the only functions for which $B^2 G = 0$ holds [5].

In this part we show that the functions of the form

$$f(\bar{z}) + z g(\bar{z}); \quad f, g \in \bar{A}$$

also have the above property.

Let us find those functions $w = u + iv$ whose deviation from being c-analytic is a c-analytic function, i. e. such functions for which

$$\bar{C}^2 w = 0. \tag{6}$$

Let $\bar{C} w = U(x, y) + iV(x, y)$

Then $\bar{C}^2 w = 0$ implies

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0; \qquad \frac{\partial V}{\partial y} - \frac{\partial U}{\partial y} = 0,$$

and, as a consequence,

$$\Delta U = 0; \quad \Delta V = 0 \tag{7}$$

Since $U = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$; $V = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$, we have

$$\frac{\partial}{\partial x}(\Delta U) + \frac{\partial}{\partial y}(\Delta V) = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4},$$

and according to (7), we get

$$\Delta^2 u = 0 \tag{8}$$

Similarly, $\Delta^2 v = 0$; which proves one part of the above statement.

Condition (6) separates into the following system

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} = 0 \\ &\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y^2} = 0. \end{aligned} \tag{9}$$

Adding and subtracting the above equations, we get

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=-2\frac{\partial^2 v}{\partial x\,\partial y}$$

(10)

$$\frac{\partial^2 v}{\partial y^2}-\frac{\partial^2 v}{\partial x^2}=-2\frac{\partial^2 u}{\partial x\,\partial y}$$

Systems (9) and (10) are equivalent.

Start from the equation (8), which in a developed form becomes

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\,\partial y^2}+\frac{\partial u^4}{\partial^4 y}=0.$$

The general solution of this equation is [6]

(11) $$u(x,y)=f(x+iy)+yg(x+iy)+\varphi(x-iy)+y\psi(x-iy)$$

where f, g, φ, ψ are arbitrary functions.

From the first equation (10), and (11), we get

$$\frac{\partial^2 v}{\partial x\,\partial y}=-f''(x+iy)+ig'(x+iy)-yg''(x+iy)-\varphi''(x-iy)-$$
$$-i\psi'(x-iy)-y\psi''(x-iy)$$

and after integration with respect to x

$$\frac{\partial v}{\partial y}=-f'(x+iy)+ig(x+iy)-yg'(x+iy)-\varphi'(x-iy)$$
$$-i\psi(x-iy)-y\psi'(x-iy)+F(y)$$

where F is an arbitrary function.

After integration with respect to y we get

(12) $$v(x,y)=if(x+iy)+iyg(x+iy)-i\varphi(x-iy)-iy\psi(x-iy)+\Psi(y)+\Phi(x)$$

where Φ is an arbitrary function, and $\Psi(y)=\int F(y)dy$.

In order to determine the functions $\Psi(y)$, $\Phi(x)$ we shall use, besides equation (12), the equation (11) and the second equation of the system (10) which becomes

$$\Psi''(y)-\Phi''(x)=0.$$

This implies

$$\Phi(x)=ax^2+bx+c,\quad \Psi(y)=ay^2+dy+e$$

where a, b, c, d, e are constants.

Therefore

$$\Phi(x)+\Psi(y)=ax^2+bx+c+ay^2+dy+e,$$

i. e. $$\Phi(x)+\Psi(y)=\left(\frac{b\bar{z}}{2}+\frac{i}{2}\,d\bar{z}+c+e\right)+z\left(a\bar{z}+\frac{b}{2}-\frac{id}{2}\right)=\alpha(\bar{z})+z\,\beta(\bar{z})$$

The function w therefore becomes

$$w=u+iv=2\,\varphi(x-iy)+2y\,\psi(x-iy)+i\,\alpha(\bar{z})+iz\,\beta(\bar{z})$$

$$=2\,\varphi(\bar{z})-iz\,\psi(\bar{z})+i\bar{z}\,\psi(\bar{z})+i\,\alpha(\bar{z})+iz\,\beta(\bar{z})=F(\bar{z})+zG(\bar{z})$$

where $F(\bar{z})=2\,\varphi(\bar{z})+i\bar{z}\,\psi(\bar{z})+i\,\alpha(\bar{z})\qquad G(\bar{z})=-i\,\psi(\bar{z})+i\,\beta(\bar{z})$.

Therefore, functions of the form $w=F(\bar{z})+zG(\bar{z})$ have the property that $\bar{C}^2w=0$, which proves the second part of the above statement.

2. Continuing this procedure it can be shown, similarly as in [7], that the functions of the form $\alpha(\bar{z})+z\,\beta(\bar{z})+z^2\,\gamma(\bar{z})$ have the property that their second deviation from being c-analytic is a c-analytic function as well as that their real and imaginary parts $u(x,y)$ and $v(x,y)$ satisfy

$$\Delta^3u=0,\quad \Delta^3v=0.$$

When examining function whose *n-th* deviation from being c-analytic is a c-analytic function and, in connection with that, functions whose real and imaginary parts satisfy

$$\Delta^nu=0,\quad \Delta^nv=0,$$

it is more convenient, following the method of S. Fempl, [8], to use the operator which is inverse to $\bar{C}$.

We shall, however, use mixed partial derivatives with respect to z, $\bar{z}$ and we shall obtain a more general result which contains Fempl's result on areolare polynomials and, also its analogue on c-areolare polynomials.

Lemma. *Let $w(z,\bar{z})=u(x,y)+iv(x,y)$ be a complex function whose partial derivatives with respect to z, $\bar{z}$ are continuous.*

Then

$$\frac{\partial^{2n}w}{\partial z^n\,\partial\bar{z}^n}=\frac{1}{2^{2n}}\,(\Delta^n u+i\,\Delta^n v) \tag{13}$$

Proof. Let $n=1$. Then

$$\frac{\partial^2 w}{\partial z\,\partial\bar{z}}=\frac{1}{4}\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right)\left(\frac{\partial w}{\partial x}-i\,\frac{\partial w}{\partial y}\right)=\frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{1}{4}\,(\Delta u+i\,\Delta v).$$

Suppose that (13) holds for some n. Then

$$\frac{\partial^{2n+2} w}{\partial z^{n+1}\,\partial \bar{z}^{n+1}} = \frac{1}{4}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\frac{1}{2^{2n}}\,(\Delta^n u + i\,\Delta^n v)$$

$$= \frac{1}{2^{2n+2}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\Delta^n u + i\,\Delta^n v)$$

$$= \frac{1}{2^{2n+2}}\,\Delta\,(\Delta^n u + i\,\Delta^n v) = \frac{1}{2^{2n+2}}\,\Delta^{n+1} u + i\,\Delta^{n+1} v),$$

and by induction, the proof is complete.

Corollary

$$\frac{\partial^{2n} w}{\partial z^n\,\partial \bar{z}^n} = 0 \text{ if and only if } \Delta^n u = 0 \text{ and } \Delta^n v = 0.$$

Theorem 7. *Complex functions of the form*

(1) $$\sum_{\nu=0}^{n-1} [\alpha_\nu(z)\,\bar{z}^\nu + \beta_\nu(\bar{z})\,z^\nu]$$

where $\alpha_i(z) \in A$, $\beta_i(\bar{z}) \in \bar{A}$ $(i = 0, 1, \ldots, n-1)$, *and only those functions, have the property that their real ant imaginary part satisfy the equations*

$$\Delta^n u = 0; \quad \Delta^n v = 0.$$

Proof. The conditions $\Delta^n u = 0$ and $\Delta^n v = 0$ are equivalent to

$$\frac{\partial^{2n} w}{\partial z^n\,\partial \bar{z}^n} = 0.$$

The general solution of the above equation is (14).

Special cases:

1. Putting $\beta_\nu(\bar{z}) = 0$, $\nu = 0, 1 \ldots, n-1$, we get the areolare polynomial. This result has been proved by S. Fempl [8]. Differentiating $n-1$ times with respect to $\bar{z}$ we see that the areolare polynomial is a non-analytic function whose $(n-1)$-*th* deviation from being analytic function is an analytic function.

2. Putting $\alpha_\nu(z) = 0$, $\nu = 0, 1, \ldots, n-1$, we get the c-areolare polynomial. Differentiating $n-1$ times with respect to z we see that its $(n-1)$—th deviation from being c-analytic is a c-analytic function.

REFERENCES

[1] А. Билимовић, *Диференцијални елементи геометријске теорије неаналитичких функција*, Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka., CCX, (1960), 1—82.

[2] Б. Риман, *Сочинения*, Москва, 1948.

[3] Lars Hörmander, *An Introduction to Complex Analysis in Several Variables* Princeton, New Jersey, 1966.

[4] В. И. Смирнов, *Курс высшей математики*, Т III, Москва, 1956.

[5] С. Фемпл, *О неаналитичким функцијама чије је одступање од аналитичности аналитичка функција*, Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka., CCLIV, 24 (1963), 75-80.

[6] A. R. Forsyth, *Lehrbuch des Differential-Gleichungen*, Braunschweig, 1912.

[7] С. Фемпл, *О неаналитичким функцијама чије је други одступање од аналитичности аналитичка функција*, Bull. Soc. Math. Phys. Serbie, XV (1963), 57-62.

[8] S. Fempl, *Aleolarni polinomi kao klasa neanalitičkih funkcija čiji su realni i imaginarni delovi poliharmonijske funkcije*, Mat. vesnik 1 (16), 1964.