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SOME REMARQUES ON HYPERGOUPS

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The aim of this note is examination of hypergroup with operators and distributivity and strong distributivity of outer operations. It is also shown that every semiring is strongly distributive hypergroup. Other questions treated in this paper are: self-distributivity and balanceness.

A hypergroup is an algebraic system satisfying all the axioms of a group except that the muliplication is multivalued.

Let us give the main definitions concerning hypergroup according to [3].

A hypergroup G is an algebraic system with one operation called *multiplication*. This operation satisfies the following axioms:

The product. If a and b are two elements of G *then* the product $a \cdot b$ is subset of G

(1) $$a \cdot b = (c_1, c_2, \ldots)$$

No assumptions on the number of elements in the product are made; it may be arbitrary and vary from product to product. The definition of a product is extended to arbitrary subsets in the following way. If

$$A = (a_1, a_2, \ldots)$$

$$B = (b_1, b_2, \ldots)$$

then

$$A \cdot B = (\ldots, a_i \cdot b_j, \ldots)$$

is the set consisting of all elements of G contained in some product $a_i \cdot b_j$.

The association law. For any three elements of G we have

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c.$$

These products have a meaning according to the definition of products of subsets.

The quotient axiom. To any two elements a and b there shall exist other elements x and y such that

$$b \in a \cdot x, \qquad b \in y \cdot a. \tag{2}$$

An element e such that $a \in e \cdot a$ for all $a \in G$ is called a *left unit*, and a *right unit* is defined analogously. A *unit* is an element e such that

$$a \in e \cdot a, \qquad a \in a \cdot e$$

for all a. If

$$e \cdot a = a \qquad (a \cdot e = a)$$

for all a, then e is called a *left* (*a right*) *scalar unit*. If

$$a \cdot e = e \cdot a = a$$

for all a then e is a *scalar unit* or *absolute unit*. We shall say that a^{-1} is a *left inverse* of a when

$$e \in a^{-1} \cdot a$$

where e is some unit element. One defines *right inverses* in a similar manner. A *two-sided inverse* a^{-1} has the properties

$$e \in a \cdot a^{-1}, \qquad e \in a^{-1} \cdot a$$

where e is some two sided unit element.

The existence of a left (right) inverse corresponding to some left (right) unit element follows from the quotient axiom. Two-sided inverses are not postulated by above axioms.

One finds easily that if every product in a hypergroup contains but one element the hypergroup is a group.

An application $f : \Omega \times E \to P(E)$ ($P(E)$ is the set of all nonempty subsets of E) is said to be an *outer multivalued operation* in the set E (see [3]). The set Ω is said to be the domain of operators.

An outer multivalued operation $\square$ is said to be *distributive* with respect to an inner multivalued operation $\circ$ in E ($a, b \in E$, $a \circ b \subset E$) if

$$\alpha \square (a \circ b) \subseteq (\alpha \square a) \circ (\alpha \square b),$$

for all $\alpha \in \Omega$ and all $a, b \in E$.

If for all $\alpha \in \Omega$ and $a, b \in E$ we have

$$\alpha \square (a \circ b) = (\alpha \square a) \circ (\alpha \square b),$$

the distributivity is said to be strong one.

A hypergroup $(E, \circ)$ equipped with a distributive outer operation $\square$ with operators $\{\alpha\} = \Omega$ is said to be a *hypergroup with operators*.

These definitions are given in [3] and some properties of hypergroups with operators are examined.

In [1] the following problem is treated. Given a group operation $*$ on the same set G, whether there exists a hypergroup operation $\circ$ on G such that $a*b \in a\circ b$ for all $a, b \in G$. If the answer is yes the hypergroup operation $\circ$ is said to *stretch* the group operation $*$. In the same way, a hypergroup operation can be given and a group operation asked for. In the case of its existence one says that the group operation *balances* the hypergroup operation $\circ$ (and hypergroup is said to be *balanceable*).

Among others the following two theorems are proved:

Theorem A. Every group operation can be stretched.

Therem B. Every hypergroup operation should not be balanced.

The manner of extension of a group operation $*$ to a hypergroup operation $\circ$, which stretches $*$ is not unique. One manner of extension of $*$ to $\circ$ (in this paper $*$ always denotes a group and $\circ$ a hypergroup operation) is as follows:

$$a\circ b = \{a*b, \quad a, b\}. \tag{R}$$

A hypergroup G is said to be *homeomorphic* [2] to another hypergroup G' when there exists a correspondence $a \to a'$ between the elements of the two systems such that, when

$$c \in a\circ b$$

then

$$c' \in a'\circ b'.$$

If $G' = G$, then this correspondence is called *endomorphism*.

Let Ω be a set of operators for a hypergroup G.

Then, $\delta(a\circ b) \subseteq (\delta a)\circ(\delta b)$, for $\delta \in \Omega$.

Let $c' \in \delta(a\circ b)$. Then there exists $c \in a\circ b$ such that $c' = \delta c$. If we put $\delta a = a'$ and $\delta b = b'$, we have

$$c' \in a'\circ b'$$

i.e. δ is endomorphism of $(G, \circ)$.

Conversely, let δ be an endomorphism of $(G, \circ)$. Then $c \in a\circ b$ implies

$$\delta c \in (\delta a)\circ(\delta b) \tag{1}$$

(1) can be written

$$c' \in a'\circ b'$$

or

$$(a\circ b)' \subseteq a'\circ b'.$$

It means that $\Omega = \{\delta\}$ is a set of operators for hypergroup with operators G.

So we have proved the following

P r o p o s i t i o n 1. *If G is a hypergroup with operators $\Omega = \{\delta\}$, then every element of Ω is an endomorphism of $(G, \circ)$. Conversely, every set of endomorphism of $(G, \circ)$ serves as a set of operators for a hypergroup with operators.*

$(G, +, \cdot)$ is a *semiring* if two binary operations are defined on the set G, addition $+$ and multiplication $\cdot$, and if a twosided law of distributivity of the multiplication with regard to the addition is satisfied:

$$a\cdot(b+c)=a\cdot b+a\cdot c, \qquad (b+c)\cdot a=b\cdot a+c\cdot a,$$

for every $a, b, c\in G$.

The notion of a semiring seems to be first introduced by Vandiver in [4].

Proposition 2. *Any semiring becomes a strongly distributive hypergroup if the hypergroup operation* $\circ$ *is defined in the manner* (R) *and the outer operation is the multiplication* $\circ$ *in the semiring.*

Proof. Put $a\circ b=\{a+b,\ a,\ b\}$. The associativity of the operation $\circ$ is easily verified:

$$\begin{aligned}(a\circ b)\circ c&=\{a+b,\ a\ b\}\circ c\\&=(a+b)\circ c\cup a\circ c\cup b\circ c\\&=\{a+b+c,\ a+b,\ a+c,\ b+c,\ a,\ b,\ c\},\end{aligned}$$

and in the same way

$$a\circ(b\circ c)=\{a+b+c,\ a+b,\ a+c,\ b+c,\ a,\ b,\ c\}.$$

For given $a, b\in G$, the required elements x and y are $x=y=b$, because of

$$a\circ b=\{a+b,\ a,\ b\}\ni b$$
$$b\circ a=\{b+a,\ b,\ a\}\ni b.$$

Since G is semiring, another inner operation $\circ$ is defined in G, which is distributive with respect to the operation $+$. So we have

$$\begin{aligned}a\cdot(b\circ c)&=a\cdot\{b+c,\ b,\ c\}\\&=\{a\cdot b+a\cdot c,\ a\cdot b,\ a\cdot c\}\end{aligned}$$

on the other hand

$$(a\cdot b)\circ(a\cdot c)=\{a\cdot b+a\cdot c,\ a\cdot b,\ a\cdot c\}$$

and the strong distributivity holds.

Proposition 3. *Let* Ω *be some set of endomorphisms of the group* $(G, *)$. *Then outer operation* $\square$, $\alpha\square a=\alpha a$ *in the hypergroup* $(G, \circ)$ *with* $\circ$ *defined by* (R), *is strongly distributive and* $(G, \circ)$ *is a hypergroup with operators.*

The only thing to be proved is the distributivity law. Denote by a' the image of a under the endomorphism $\alpha \in \Omega$. Then one has

$$\begin{aligned}\alpha\square(a\circ b) &= \{\alpha(a*b),\ \alpha a,\ \alpha b\}\\ &= \{(\alpha a)*(\alpha b),\ \alpha a,\ \alpha b\}\\ &= \{a'*b',\ a',\ b'\}.\end{aligned}$$

since α is an endomorphism in $(G, *)$. But right hand side is just $a'\circ b'$ and the equality

$$\alpha\square(a\circ b)=(\alpha\square a\circ(\alpha\square b)$$

holds.

Definition. A hypergroup operation $\circ$ *on* G *is said to be self distributive if*

$$(\alpha)\qquad (a\circ b)\circ c=(a\circ c)\circ(b\circ c),\quad \text{and}$$

$$(\beta)\qquad a\circ(b\circ c)=(a\circ b)\circ(a\circ c)$$

No group operation on a set with more than one element is self distributive.

Proposition 4 *The hypergroup* G *defined in proposition* 3 *of* [3] *is self distributive.*

Proof. If at least one of a, b and c is e, the both sides of (α) and (β) equal G, according to the conditions 1° and 2° of the proposition 3 in [3], and consequently (α) and (β) hold. Suppose that $a\neq e$, $b\neq e\neq c$. Then, according to the condition 1° of proposition 3 in [3], $e\in a\circ b$ and so $(a\circ b)\circ(a\circ c)=G$ in virtue of the condition 2° in [3]. The same reason can be applied to the left hand side of (β) and also for (α), so that (α) and (β) hold.

Proposition 5. *Any partition of a set* X *implies a hypergroup operation on* X.

Proof. Let $\mathcal{P}=\{P\}$ be a partition of X. Denote by $[a]$ the equivalence class to which a belongs. Define a multivalued operation on X in the following way:

$$a\mathring{\circ} b=[a]\cup[b].$$

Let us prove that the operation $\circ$ is associative one. Consider

$$A=a\circ(b\circ c).$$

According to the definition of $\circ$,

$$A=a\circ([b]\cup[c]).$$

Since $a\circ m=[a]\cup[m]$ for all $m\in[b]$ and $a\circ n=[a]\cup[c]$ for all $n\in[c]$, one concludes

$A=[a]\cup[b]\cup[c]$, (operation $\cup$ being associative).

For the same reasons the expression $B=(a\circ b)\circ c$ obtains the same value, and the associativity is proved.

To finish the proof of theorem take an arbitrary pair $(a, b) \in G \times G$. Then evidently $a \circ b \ni b$ and $b \circ a \ni b$ and the quotient law is fulfilled.

A hypergroup of this type we shall call *partition type hypergroup*. A partition of X is said to be trivial if $\mathcal{P} = \{X\}$.

A hypergroup is trivial if $a \circ b = X$ for all $a, b \in X$.

Proposition 6. *No nontrivial partition type hypergroup is balanceable.*

Proof. Suppose on the contrary that there exists a group operation $*$ on X, which balances the operation $\circ$. Then $(X, *)$ has unity. Denote it by e. Let $a \in X$ be arbitrary. With above supposition there exists a^{-1} such that $a * a^{-1} = e$. According to the definition of balanceness, it means that $e \in [a] \cup [a^{-1}]$, and hence $[e] = X$, since a is arbitrary. This contradicts the hypothesis of the theorem that partition is nontrivial.

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