

# CHARACTERISATION OF SOME ORTHOGONAL POLYNOMIALS

by

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(Received January 22, 1968)

§ 1. Let  $P_n^{(\alpha, \beta)}(x)$  be the  $n^{\text{th}}$  Jacobi polynomial. The following generating function of the Jacobi polynomials is well known [1]

$$(1) \quad (1-t)^{-1-\alpha-\beta} {}_2F_1 \left[ \begin{matrix} \frac{1}{2} (1+\alpha+\beta), & \frac{1}{2} (2+\alpha+\beta); & \frac{2t(x-1)}{(1-t^2)^2} \\ 1+\alpha \end{matrix} \right] \\ = \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n} P_n^{(\alpha, \beta)}(x) t^n.$$

This suggests the consideration of the class of polynomial sets  $\{P_n(x); n=0, 1, \dots\}$ ,  $P_n(x)$  is of exact degree  $n$  and

$$(2) \quad (1-t)^{-c} \Phi \left( \frac{2t(x-1)}{(1-t)^2} \right) = \sum_{n=0}^{\infty} \frac{(c)_n}{(c-\beta)_n} P_n(x) t^n$$

holds where  $\Phi(u)$  has a formal power series expansion,  $\Phi(0) \neq 0$ .

It is obvious that the set of Jacobi polynomials is only one of many possible sets in the above class. It is therefore of ample interest to find what else is required in order to characterise the Jacobi polynomials by means of (1.2). We give below two such characterisation. We show that either  $\{P_n(x)\}$  be orthogonal or a hypergeometric function of certain type in order that  $P_n(x)$  be essentially the  $n^{\text{th}}$  Jacobi polynomial.

§ 2. Differentiating (1.2) with respect to  $x$  and  $t$  respectively, we get

$$(1) \quad 2t(1-t)^{-c-2} \Phi' \left( \frac{2t(x-1)}{(1-t)^2} \right) = \sum_{n=0}^{\infty} \frac{(c)_n}{(c-\beta)_n} P'_n(x) t^n$$

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\* Supported by a post doctoral fellowship of the University of Alberta, Edmonton, Canada.

and

$$c(1-t)^{-c-1} \Phi \left( \frac{2t(x-1)}{(1-t)^2} \right) + 2(1-t)^{-c-3}(1+t)(x-1) \Phi' \left( \frac{2t(x-1)}{(1-t)^2} \right) \\ = \sum_{n=0}^{\infty} \frac{(n+1)(c)_{n+1}}{(c-\beta)_{n+1}} P_{n+1}(x) t^n.$$

Eliminating  $\Phi'$  from (1) and (2) we have on equalling like powers of  $t$

$$(3) \quad (c+n) \{(1+n) P_{n+1}(x) - (c-\beta+n) P_n(x)\} \\ = (x-1) \{(c+n) P'_{n+1}(x) + (c-\beta+n) P'_n(x)\}.$$

Now let

$$(4) \quad P_n(x) = \sum_{k=0}^n c(n, k) \left( \frac{x-1}{2} \right)^{n-k}.$$

Substituting, for  $P_n(x)$  from (4) in (3) and equating the like powers of  $\frac{x-1}{2}$ , we have

$$(5) \quad c(n+1, k+1) = \frac{(c-\beta+n)(c+2n-k)}{(c+n)(1+k)} c(n, k).$$

Iterating (5),  $n$  times, we get

$$(6) \quad c(n, k) = \frac{(c-\beta+n-k)_k (c+2n-2k)}{k! (c+n-k)_k} c(n-k, 0).$$

Now assume that the  $\{P_n(x)\}$  is a simple set of orthogonal polynomials and therefore it shall satisfy a relation of the type

$$(7) \quad P_{n+1}(x) = (A_n x + B_n) P_n(x) + C_n P_{n-1}(x),$$

where  $A_n c_n > 0$ , which can be written as

$$(8) \quad c(n+1, k+1) = 2A_n c(n, k+1) + (A_n + B_n) c(n, k) + C_n c(n-1, k-1) \\ \text{for } -1 \leq k \leq n.$$

But we have from (1.2)

$$(9) \quad P_n(1) = \frac{(c-\beta)_n}{(1)_n} \Phi(0).$$

For  $x=1$ , (7) gives on using (9)

$$(10) \quad A_n + B_n = \frac{c-\beta+n}{n+1} - \frac{n c_n}{c-\beta+n-1}.$$

But for  $K=-1$  and 0, (8) gives

$$(11) \quad A_n = \frac{c(n+1, 0)}{2 c(n, 0)}$$

and

$$(12) \quad c(n+1, 1) = 2A_n c(n, 1) + (A_n + B_n) c(n, 0).$$

Substituting for  $c$ 's from (6) in (12) we have

$$(13) \quad A_n + B_n = \frac{(c-\beta+n)(c+2n)}{c+n} \cdot \frac{A_n}{A_{n-1}} \cdot \frac{(c-\beta+n+1)(c+2n-2)}{(c+n-1)}.$$

Since  $A_n \neq 0$  therefore (10) and (13) give

$$C_n = \frac{(c-\beta+n)(c-\beta+n-1)}{n(n+1)} \cdot \frac{(c-\beta+n)(c-\beta+n-1)(c+2n)}{n(c+n)} \\ + \frac{A_n}{A_{n-1}} \cdot \frac{(c-\beta+n-1)^2(c+2n-2)}{n(c+n-1)}.$$

Now (8) for  $k=1$ , with the help of (6) and (10), (12), gives

$$\frac{1}{A_n} \left[ \frac{(c+\beta+n)(c+2n-2)(c+2n-1)}{2(c+n-1)(c+n)} - \frac{(c-\beta+n)(c+2n-2)(c+2n)}{(c+n)(c+n-1)} \right. \\ \left. - \frac{(c-\beta+n)}{n(n+1)} + \frac{(c-\beta+n)(c+2n)}{n(n+c)} \right] \\ = \frac{1}{A_{n-1}} \left[ \frac{(c-\beta+n-1)(c+2n-2)}{n(c+n-1)} - \frac{(c-\beta+n-1)(c+2n-2)^2}{(c+n-1)^2} \right] \\ + \frac{1}{A_{n-2}} \left[ \frac{(c-\beta+n-2)(c+2n-4)(c+2n-3)}{2(c+n-1)(c+n-2)} \right].$$

Now set

$$A_n = \frac{(c+2n+1)(c+2n)}{2(n+1)(c+n)(c-\beta+n)} \cdot \frac{1}{D_n},$$

so that (14) becomes

$$D_n = 2D_{n-1} - D_{n-2}.$$

Iterating it  $n$  times, we have

$$D_n = (D_1 - D_0)n + D_0,$$

where  $D_0$  and  $D_1$  are constants independent of  $n$  and  $x$ , with  $D_0 \neq 0$ . Now three cases arise.

Case I. If  $D_1 \neq 0$ ,  $D_0 \neq 0$ ,  $D_1 \neq D_0$  then  $D_n = a(n+b)$ , and we have

$$P_n(x) = A \frac{(c)_n (c-\beta)_n}{(b)_n} P_n^{(b-1, c-b)} \left( \frac{x+a-1}{a} \right)$$

where  $A$  is some constant.

Case II. If  $D_1 = D_0 \neq 0$ , then  $D_n = D_0$  and we have

$$P_n(x) = \frac{\beta (c-\beta)_n}{(1)_n} Y_n^{(c-1)}(1-xD_0)$$

where  $B$  is some constant and  $Y_n^{(\alpha)}(x)$  is a Bessell polynomial defined as

$$Y_n^{(\alpha)}(x) = {}_2F_0 \left[ -n; n + \alpha + 1; -\frac{x}{2} \right]$$

C a s e III. If  $D_1 = 0$  then  $D_n = (1-n)D_0$  and we have

$$P_n(x) = E \frac{(c-\beta)_n}{(1)_{n-2}} P_n^{(-2, c+1)} \left( \frac{1+D_0-x}{D_0} \right)$$

where  $E$  is some constant.

**Theorem 1.** *The simple set of polynomials  $\{P_n(x)\}$  where  $\deg P_n(x) = n$ , which is orthogonal and satisfies (1.2) is either the set of Jacobi or Bessel polynomials.*

§3. In this section we assume that the polynomial set  $\{P_n(x)\}$  of (1.2) satisfies

$$(1) \quad P_n(x) = \frac{(c)_{2n}}{n!(c)_n} \left( \frac{x-1}{2} \right)^n {}_pF_q \left[ \begin{matrix} -n, 1-\alpha-c-n, (\alpha_{p-2}); \\ 1-c-2n, (\beta_{q-1}) \end{matrix}; \frac{2}{1-x} \right]$$

where the parameters  $c, \alpha, \alpha_1, \dots, \alpha_{p-2}, \beta_1, \dots, \beta_{q-1}$  are arbitrary complex numbers with  $\beta_k \neq -m$  ( $a$  negative integer).

Setting  $P_n(x) = \sum_{k=0}^n c(n, k) \left( \frac{x-1}{2} \right)^{n-k}$  in (1.2) and then equating the coefficients of powers of  $\left( \frac{x-1}{2} \right)$ , we get

$$(2) \quad c(n+1, k+1) = \frac{(c-\beta+n)(c+2n-k)}{(c+n)(1+k)} c(n, k).$$

Putting

$$c(n, k) = \frac{(-1)^k (c)_{2n} (-n)_k (1+\beta-c-n)_k [(\alpha_{p-2})]_k}{(1)_k (1)_n (c)_n (1-c-2n)_k [(\beta_{q-1})]_k}$$

we have

$$\frac{[(\alpha_{p-2}) + k]}{[(\beta_{q-1}) + k]} = 1,$$

and consequently we have proved

**Theorem 2.** *The only hypergeometric polynomials of the type (3.1) which has a generating function (1.2) is the set of Jacobi polynomials.*

## REFERENCES

- [1] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. revised edition 23 (1959) New York.

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