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ON THE CONVERGENCE OF CERTAIN SEQUENCES

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In [1] and [2] some theorems on the convergence of certain sequences in a complete metric space were proved. In this paper we prove a somewhat more general theorem on the convergence of sequences and we give a number of examples and corollaries.

Let E be a complete metric space and let $f_n\colon E^k \to E$ be a sequence of functions such that

$$(1)\quad d(f_n(u_1, u_2, \ldots, u_k), f_{n-1}(u_2, u_3, \ldots, u_{k+1})) \leqslant q_1 d(u_1, u_2) + q_2 d(u_2, u_3) + \cdots + q_k d(u_k, u_{k+1}) + a_{n-1}$$

for every $u_1, u_2, \ldots, u_{k+1} \in E$, where $q_1, q_2, \ldots, q_k$ are fixed nonnegative numbers such that $q_1 + q_2 + \cdots + q_k \leqslant q < 1$, whereas the series $\sum_{\nu=0}^{\infty} a_\nu$ converges $(a_\nu \geqslant 0)$.

Theorem. *Let*

$$x_{n+k} = f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}), \qquad (n = 1, 2, \ldots)$$

where the elements $x_1, x_2, \ldots, x_k$ are arbitrarily chosen. If the condition (1) *is satisfied, then:*

(i) *the sequence of functions $f_n(u, u, \ldots, u)$ converges uniformly to a function $f(u, u, \ldots, u)$;*

(ii) the sequence (x_n) converges in E;

(iii) the equation

$$x = f(x, x, \ldots, x)$$

has a unique solution $x = \lim_{n\to\infty} x_n$.

Proof.

(i) Putting $u_1 = u_2 = \cdots = u_{k+1} = u$ in (1) we get

$$d(f_n(u, u, \ldots, u), f_{n-1}(u, u, \ldots, u)) \leqslant a_{n-1}$$

from which it can easily be seen that the sequence of functions $f_n(u, u, \ldots, u)$ converges uniformly to a function $f(u, u, \ldots, u)$.

(ii) Denote $d(x_n, x_{n+1})$ by Δ_n. We then obtain the following system of inequalities

$$\Delta_{n+k+i} \leqslant a_{n+i} + q_1 \Delta_{n+i} + q_2 \Delta_{n+1+i} + \cdots + q_k \Delta_{n+k-1+i}\ (\mathrm{i} = 0, 1, \ldots, s).$$

Adding together the above inequalities we get

$$\sum_{\nu=0}^{s} \Delta_{n+k+\nu} \leqslant \sum_{\nu=n}^{n+s} a_\nu + q \sum_{\nu=0}^{s} \Delta_{n+k+\nu} + q\,(\Delta_n + \Delta_{n+1} + \cdots + \Delta_{n+k-1})$$

i.e.

$$\sum_{\nu=0}^{s} \Delta_{n+k+\nu} \leqslant \frac{q}{1-q} \sum_{\nu=n}^{n+s} a_\nu + \frac{q}{1-q}(\Delta_n + \Delta_{n+1} + \cdots + \Delta_{n+k-1}) \tag{2}$$

Since

$$\limsup_{n\to\infty} \Delta_{n+k} \leqslant q_1 \limsup_{n\to\infty} \Delta_n + \cdots + q_k \limsup_{n\to\infty} \Delta_{n+k-1}$$

i.e.

$$\limsup_{n\to\infty} \Delta_n \leqslant q \limsup_{n\to\infty} \Delta_n$$

we conclude that

$$\lim_{n\to\infty} \Delta_n = 0$$

Letting $n \to \infty$ in (2) we obtain:

$$d(x_{n+k}, x_{n+k+s}) \leqslant \sum_{\nu=0}^{s} \Delta_{n+k+\nu} \to 0$$

which means that (x_n) is a *Cauchy*'s sequence. Since E is complete, (x_n) converges, i.e. $\lim_{n\to\infty} x_n = x$.

(iii)

$$\begin{aligned}
d(x_{n+k}, f(x, x, \ldots, x)) &= d(f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x, x, \ldots, x)) \\
&\leqslant d(f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}), f_{n+1}(x, x_n, \ldots, x_{n+k-2})) \\
&\quad + d(f_{n+1}(x, x_n, \ldots, x_{n+k-2}), f_{n+2}(x, x, \ldots, x_{n+k-3})) \\
&\quad + \cdots \\
&\quad + d(f_{n+k-1}(x, x, \ldots, x_n), f_{n+k}(x, x, \ldots, x)) \\
&\quad + d(f_{n+k}(x, x, \ldots, x), f(x, x, \ldots, x)) \\
&\leqslant \sum_{\nu=n}^{n+k-1} a_\nu + q\,(d(x, x_n) + d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k-2})) \\
&\qquad + d(f_{n+k}(x, x, \ldots, x), f(x, x, \ldots, x)) \to 0
\end{aligned}$$

when $n \to \infty$.

Therefore,

$$x = \lim_{n\to\infty} x_n = f(x, x, \ldots, x).$$

Furthermore,

$$\begin{aligned}
d(f(u, u, \ldots, u),\ & f(v, v, \ldots, v)) \\
&\leqslant d(f(u, u, \ldots, u),\ f_n(u, u, \ldots, u)) \\
&\quad + d(f_n(u, u, \ldots, u), f_{n+1}(v, u, , \ldots, u)) \\
&\quad + d(f_{n+1}(v, u, \ldots, u), f_{n+2}(v, v, \ldots, u)) \\
&\quad + \cdots \\
&\quad + d(f_{n+k}(v, v, \ldots, v),\ f(v, v, \ldots, v)) \\
&\leqslant (q_1 + q_2 + \cdots + q_k)\, d(u, v) + d(f(u, u, \ldots, u),\ f_n(u, u, \ldots u)) \\
&\quad + d(f_{n+k}(v, v, \ldots, v),\ f(v, v, \ldots, v)) + \sum_{\nu=n}^{n+k-1} a_\nu
\end{aligned}$$

Letting $n \to \infty$, we get

$$d(f(u,u,\ldots,u),\ f(v,v,\ldots,v)) \leqslant (q_1+q_2+\cdots+q_k)\,d(u,v).$$

The uniqueness of the solution follows then from the *Banach fixed-point theorem* applied to $F(u)=f(u,u,\ldots,u)$.

Examples and corollaries.

Example 1.

Let the sequence (x_n) be defined by the equality

$$x_{n+2}+ax_{n+1}+bx_n=\varphi(n) \qquad (n=1,\ 2,\ldots)$$

where $x_1,\ x_2$ are arbitrarily chosen, and let

(a) $$|a|+|b|\leqslant q<1$$

(b) the series $\sum_{n=1}^{\infty}|\varphi(n)-\varphi(n-1)|$ be convergent.

It can easily be seen that the sequence of functions $f_n(x,y)=-ax-by+\varphi(n)$ satisfies the condition of the theorem, and therefore the sequence (x_n) converges.

Proposition 1. (*d'Alembert*)

A sufficient condition for the convergence of the series

$$\sum_{\nu=0}^{\infty} b_\nu\ (b_\nu>0) \text{ is } \frac{b_n}{b_{n-1}}\leqslant q<1.$$

Let $$s_n=\sum_{\nu=0}^{n-1} b_\nu.$$

Then $$s_{n+1}=s_n+b_n,$$

i.e. $$s_{n+1}=f_n(s_n),\ \text{where } f_n(u)=u+b_n.$$

The sequence (s_n) will converge if the functions $f_n(u)$ satisfy (1). We choose $a_n=0$.

(1) then becomes:

$$|s_n+b_n-s_{n-1}-b_{n-1}|\leqslant q\,|s_n-s_{n-1}|,\quad 0<q<1$$

i.e. $$\frac{b_n}{b_{n-1}}\leqslant q\leqslant 1.$$

Proposition 2.

Let E be the set of real numbers and let $h(v_1,v_2,\ldots,v_{k+1})$ be a function whose partial derivatives exist and satisfy the following conditions:

$$\sum_{i=1}^{k}\left|\frac{\partial h}{\partial v_i}\right|\leqslant q<1$$

$$\sum_{i=0}^{\infty}\left|\frac{\partial h}{\partial v_{k+1}}\right|_{v_{k+1}=i} \text{ converges.}$$

Put $f_n(u_1, u_2, \ldots, u_k) = h(u_1, u_2, \ldots, u_k, n)$
Then according to the mean value theorem

$$\begin{aligned}
&|f_n(u_1, u_2, \ldots, u_k) - f_{n-1}(u_2, u_3, \ldots, u_{k+1})| \\
&= |h(u_1, \ldots, u_k, n) - h(u_2, \ldots, u_{k+1}, n-1)| \\
&\leqslant \left|\frac{\partial h}{\partial v_1}\right| |u_1 - u_2| + \cdots + \left|\frac{\partial h}{\partial v_k}\right| |u_k - u_{k+1}| + \left|\frac{\partial h}{\partial v_{k+1}}\right|
\end{aligned}$$

and therefore the sequence of functions $f_n(u_1, u_2, \ldots, u_k)$ satisfies the condition (1) which means that $f_n(u, u, \ldots, u)$ converges uniformly to $f(u, u, \ldots, u)$ and that every sequence (x_n) defined by

$$x_{n+k} = f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}) = h(x_n, x_{n+1}, \ldots, x_{n+k-1}, n) \qquad (n = 1, 2, \ldots)$$

is convergent and its limit is the unique solution of the equation

$$x = f(x, x, \ldots, x).$$

Proposition 3. (*Marjanović-Prešić*) [2].

Let E be a complete metric space and let $f_n : E^k \rightarrow E$ be a sequence of functions such that:

$$\begin{aligned}
(3) \qquad & d(f_n(u_1, u_2, \ldots, u_k), f_n(u_2, u_3, \ldots, u_{k+1})) \leqslant q_1 d(u_1, u_2) \\
& + q_2 d(u_2, u_3) + \cdots + q_k d(u_k, u_{k+1})
\end{aligned}$$

$$(4) \qquad d(f_{n+1}(u_1, u_2, \ldots, u_k), f_n(u_1, u_2, \ldots, u_k)) \leqslant a_n \quad (n = 1, 2, \ldots)$$

where $q_1, q_2, \ldots, q_k$ are fixed non-negative numbers such that $q_1 + q_2 + \cdots + q_k \leqslant q < 1$ and the series $\sum_{\nu=0}^{\infty} a_\nu$ $(a_\nu > 0)$ is convergent.

Clearly, conditions (3), (4) imply the condition (1) and therefore every sequence defined by

$$x_{n+k} = f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}) \qquad (n = 1, 2, \ldots)$$

where the functions f_n satisfy (3) and (4) is convergent and $\lim_{n \to \infty} x_n$ is the unique solution of the equation $x = f(x, x, \ldots, x)$ where $f(x, x, \ldots, x)$ is the function to which the sequence of functions $f_n(x, x, \ldots, x)$ uniformly converges.

Proposition 4.

Let R be the set of real numbers and let $f : R^{k+1} \rightarrow R$ be a function such that

$$\begin{aligned}
(5) \qquad & |f(u_1, u_2, \ldots, u_k, x) - f(u_2, u_3, \ldots, u_{k+1}, y)| \\
& \leqslant q_1 |u_1 - u_2| + \cdots + q_k |u_k - u_{k+1}| + |x - y|
\end{aligned}$$

where $q_1, q_2, \ldots, q_k$ are non-negative fixed numbers such that $q_1 + q_2 + \cdots + q_k \leqslant q < 1$.

Let the sequence (x_n) satisfy the condition

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}, b_n) \qquad (n = 1, 2, \ldots)$$

(elements $x_1, x_2, \ldots, x_k$ are arbitrarily chosen), where

(a) f satisfies (5)

(b) the sequence (b_n) is monotonic and $\lim\limits_{n\to\infty} b_n = A$ (A is a finite number).

Then: (i) the sequence (x_n) converges in R

(ii) the equation $x = f(x, x, \ldots, x, A)$ has the unique solution $x = \lim\limits_{n\to\infty} x_n$.

Putting $g_n(u_1, u_2, \ldots, u_k) = f(u_1, u_2, \ldots, u_k, b_n)$ it can easily be seen that the sequence of functions g_n satisfies the condition (1) and that, therefore, $g_n(u, u, \ldots, u)$ converges uniformly to $g(u, u, \ldots, u) = f(u, u, \ldots, u, A)$, which means that the sequence (x_n) has properties (i) and (ii).

Example 2.

The sequence (x_n) given by

$$x_{n+2} = \frac{1}{3} x_n + \frac{1}{2} x_{n+1} + 1 + \frac{1}{n} \qquad (n = 1, 2, \ldots)$$

converges to 6.

Clearly, the function $f(x, y, z) = \frac{1}{3} x + \frac{1}{2} y + z$ satisfies the condition (5) whereas the sequence $b_n = 1 + \frac{1}{n}$ is monotonic and $\lim\limits_{n\to\infty} b_n = 1$.

Proposition 5.

Let E be a complete metric space and let g_n: $E^{k+1} \to E$ be a sequence of functions such that

$$\begin{aligned} (6) \qquad & d(g_n(u_1, u_1, u_2, \ldots, u_k),\ g_{n-1}(u_2, u_2, u_3, \ldots u_{k+1})) \\ & \leqslant q_1 d(u_1, u_2) + q_2 d(u_2, u_3) + \cdots + q_k d(u_k, u_{k+1}) + a_{n-1} \end{aligned}$$

where $q_1, q_2, \ldots, q_k$ are fixed non-negative numbers such that $q_1 + q_2 + \cdots + q_k \leqslant q < 1$ and the series $\sum\limits_{\nu=0}^{\infty} a_\nu$ $(a_\nu \geqslant 0)$ is convergent.

Furthermore, let f_n: $E^n \to E$ be a sequence of functions defined by:

$$\begin{aligned} (7) \qquad & f_{n+k}(u_1, u_2, \ldots, u_n, \ldots, u_{n+k}) \\ & = g_n(u_{n+k}, f_{n+k-1}(u_1, \ldots, u_{n+k-1}), \ldots, f_n(u_1, \ldots, u_n)) \end{aligned}$$

$(n = 1, 2, \ldots)$ where the functions $f_1(u_1)$, $f_2(u_1, u_2), \ldots, f_k(u_1, u_2, \ldots, u_k)$ are arbitrarily chosen.

Let the sequence (x_n) be defined by the following equality:

$$x_{n+k} = f_{n+k-1}(x_1, x_2, \ldots, x_{n+k-1}) \qquad (n = 1, 2, \ldots)$$

(the element x_1 is arbitrarily chosen).

If the conditions (6) and (7) hold, then:

(i) the sequence of functions $g_n(u, u, \ldots, u)$ converges uniformly to a function $g(u, u, \ldots, u)$

(ii) the sequence (x_n) converges in E

(iii) the equation $x = g(x, x, \ldots, x)$ has the unique solution $x = \lim\limits_{n \to \infty} x_n$.

Denote $g_n(u_1, u_1, u_2, \ldots, u_k)$ by $G_n(u_1, u_z, u_k)$. Then the functions G_n satisfy (1) and the sequence (x_n) is defined by the relation:

$$x_{n+k} = G_n(x_{n+k-1}, x_{n+k-2}, \ldots, x_n),$$

and, therefore, according to the Theorem, the above proposition is true.

Example 3.

Let

$$x_{n+1} = x_1 - \frac{1}{2}(x_2 + x_3 + \cdots + x_n) + \frac{x_2}{2!} + \frac{x_3}{3!} + \cdots + \frac{x_n}{n!}$$

Then:

$$f_1(x_1) = x_1$$

$$f_{n+1}(x_1, x_2, \ldots, x_{n+1}) = g_{n+1}(x_{n+1}, f_n(x_1, x_2, \ldots x_n))$$

and $$x_{n+1} = f_n(x_1, x_2, \ldots, x_n)$$

where $$g_n(x, y) = y - \left(\frac{1}{2} - \frac{1}{n!}\right)x$$

The sequence (x_n) converges to 0, since

$$|g_n(x, x) - g_{n-1}(y, y)| \leqslant \frac{1}{2}|x - y| + \left|\frac{1}{n!}x - \frac{1}{(n-1)!}y\right|$$

and 0 is the only solution of the equation $x = \frac{1}{2}x$.

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