

## SOME FORMULAE INVOLVING APPELL'S FUNCTION $F_4$

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The object of this paper is to prove the following two formulae:

$$(1.1) \quad F_c(1+\alpha+\beta, 1+\alpha; 1+\beta, 1+\alpha, 1+\alpha; x, y, z) \\ = (1+x-y-z)^{-1-\alpha-\beta} F_4\left[\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta, 1 + \frac{1}{2}\alpha + \frac{1}{2}\beta; 1+\alpha, 1+\beta; \frac{4yz}{(1+x-y-z)^2}, \frac{4x}{(1+x-y-z)^2}\right]$$

and

$$(1.2) \quad \sum_{r=0}^{\infty} \frac{(\lambda)_r}{(\alpha)_r} \left(\frac{1}{2}x\right)^r F_c\left(\frac{1}{2}\lambda + \frac{1}{2}r, \frac{1}{2}\lambda + \frac{1}{2}r + \frac{1}{2}; 1+\alpha+r, \beta, \gamma; x^2, y, z\right) C_r^\alpha(t) \\ = (1-xt)^{-\lambda} F_4\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; \beta, \gamma; \frac{y}{(1-xt)^2}, \frac{z}{(1-xt)^2}\right],$$

where  $F_c$  is a hypergeometric function of three variables defined by

$$F_c(a, b; c, d, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{m+n+p} (b)_{m+n+p}}{m! n! p! (c)_m (d)_n (e)_p} x^m y^n z^p,$$

while  $F_4$  is a function of two variables defined by

$$F_4(a, b; c, d; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{m! n! (c)_m (d)_n} x^m y^n.$$

For proving (1.1), we mainly depend upon [2, p. 60,264]

$$(1.3) \quad {}_2F_1(a, b; c; x) = (1-x)^{-a} {}_2F_1\left(a, c-b; c; \frac{-x}{1-x}\right)$$

and

$$(1.4) \quad F_4[a, b; c, 1+a+b-c; x(1-y), y(1-x)] \\ = {}_2F_1(a, b; c; x) {}_2F_1(a, b; 1+a+b-c; y)$$

whereas for (1.2) we require [2, p. 283]

$$(1.5) \quad \frac{(2x)^n}{n!} = \sum_{k=0}^{[n/2]} \frac{(\alpha+n-2k)}{k! (\alpha)_{n+1-k}} c_{n-2k}^{\alpha}(x).$$

2. Starting with the left hand side of (1.1), we have

$$\begin{aligned} & F_c(1+\alpha+\beta, 1+\alpha; 1+\beta, 1+\alpha, 1+\alpha; x, y, z) \\ &= \sum_{m, n=0}^{\infty} \frac{(1+\alpha+\beta)_{m+n} (1+\alpha)_{m+n}}{m! n! (1+\beta)_m (1+\alpha)_n} x^m y^n \\ & \quad \times {}_2F_1(1+\alpha+\beta+m+n, 1+\alpha+m+n; 1+\alpha; z) \\ &= (1-z)^{-1-\alpha-\beta} \sum_{m, n=0}^{\infty} \frac{(1+\alpha+\beta)_{m+n} (1+\alpha)_{m+n}}{m! n! (1+\beta)_m (1+\alpha)_n} \left(\frac{x}{1-z}\right)^m \left(\frac{y}{1-z}\right)^n \\ & \quad \times {}_2F_1\left(-m-n, 1+\alpha+\beta+m+n; 1+\alpha; \frac{z}{z-1}\right) \quad \text{by (1.3)} \\ &= (1-z)^{-1-\alpha-\beta} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{n!} \left(\frac{y}{1-z}\right)^n {}_2F_1\left(-n, -\alpha-n; 1+\beta; \frac{x}{y}\right) \\ & \quad \times {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{z}{z-1}\right) \\ &= (1-z)^{-1-\alpha-\beta} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{n!} \left(\frac{y-x}{1-z}\right)^n {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\beta; \left(\frac{x}{x-y}\right)\right) \\ & \quad \times {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{z}{z-1}\right) \\ &= (1-z)^{-1-\alpha-\beta} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{n!} \left(\frac{y-x}{1-z}\right)^n \\ & \quad \times F_4\left(-n, 1+\alpha+\beta+n; 1+\alpha, 1+\beta; \frac{yz}{(x-y)(1-z)}, \frac{x}{(x-y)(1-z)}\right) \quad \text{by (1.4)} \\ &= (1-z)^{-1-\alpha-\beta} \sum_{n=0}^{\infty} \sum_{p+q \leq n} \frac{(1+\alpha+\beta)_{n+p+q}}{(n-p-q)! p! q! (1+\alpha)_p (1+\beta)_q} \\ & \quad \times \left(\frac{y-x}{1-z}\right)^n \left[\frac{-yz}{(x-y)(1-z)}\right]^p \left[\frac{x}{(x-y)(1-z)}\right]^q \\ &= (1-z)^{-1-\alpha-\beta} \sum_{n, p, q=0}^{\infty} \frac{(1+\alpha+\beta)_{n+2p+2q}}{n! p! q! (1+\alpha)_p (1+\beta)_q} \left(\frac{y-x}{1-z}\right)^n \left[\frac{yz}{(1-z)^2}\right]^p \left[\frac{x}{(1-z)^2}\right]^q \\ &= (1+x-y-z)^{-1-\alpha-\beta} F_4\left[\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta, 1 + \frac{1}{2}\alpha + \frac{1}{2}\beta; 1+\alpha, 1+\beta; \frac{4yz}{(1+x-y-z)^2}, \frac{4x}{(1+x-y-z)^2}\right]. \end{aligned}$$

This completes the proof of (1.1).

For  $x=0$ , it becomes

$$(2.1) \quad F_4(1+\alpha+\beta, 1+\alpha; 1+\alpha, 1+\alpha; y, z) = (1-y-z)^{-1-\alpha-\beta} \\ \times {}_2F_1\left[\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta, 1 + \frac{1}{2}\alpha + \frac{1}{2}\beta; 1+\alpha; \frac{4yz}{(1-y-z)^2}\right],$$

while for  $z=0$ , it yields

$$(2.2) \quad F_4(1+\alpha+\beta, 1+\alpha; 1+\beta, 1+\alpha; x, y) = (1+x-y)^{-1-\alpha-\beta} \\ \times {}_2F_1\left[\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta, 1 + \frac{1}{2}\alpha + \frac{1}{2}\beta; 1+\beta; \frac{4x}{(1+x-y)^2}\right].$$

3. To prove (1.2), we consider

$$(1-xt)^{-\lambda} F_4\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; \beta, \gamma; \frac{y}{(1-xt)^2}, \frac{z}{(1-xt)^2}\right] \\ = \sum_{p, q, r=0}^{\infty} \frac{(\lambda)_{2p+2q+r}}{p! q! (\beta)_p (\gamma)_q} \left(\frac{1}{4}y\right)^p \left(\frac{1}{4}z\right)^q \left(\frac{1}{2}x\right)^r \frac{(2t)^r}{r!} \\ = \sum_{p, q, r=0}^{\infty} \frac{(\lambda)_{2p+2q+r}}{p! q! (\beta)_p (\gamma)_q} \left(\frac{1}{4}y\right)^p \left(\frac{1}{4}z\right)^q \left(\frac{1}{2}x\right)^r \sum_{k=0}^{[r/2]} \frac{(\alpha+r-2k)}{k! (\alpha)_{r+1-k}} C_{r-2k}^{\alpha}(t) \\ \text{by (1.5)} \\ = \sum_{r, k, p, q=0}^{\infty} \frac{(\alpha+r)(\lambda)_{r+2k+2p+2q}}{k! p! q! (\alpha)_{r+k+1} (\beta)_p (\gamma)_q} \left(\frac{1}{2}x\right)^{r+2k} \left(\frac{1}{4}y\right)^p \left(\frac{1}{4}z\right)^q C_r^{\alpha}(t) \\ = \sum_{r=0}^{\infty} \frac{(\lambda)_r}{(\alpha)_r} \left(\frac{1}{2}x\right)^r F_4\left[\frac{1}{2}\lambda + \frac{1}{2}r, \frac{1}{2}\lambda + \frac{1}{2}r + \frac{1}{2}; \right. \\ \left. 1+\alpha+r, \beta, \gamma; x^2, y, z\right] C_r^{\alpha}(t),$$

which proves (1.2).

Now we put  $z=0$ ,  $\beta=\lambda-\alpha+\frac{1}{2}$  and replace  $x$  and  $y$  by  $\sqrt{x(1-y)}$  and  $y(1-x)$  respectively. Thus using (1.4), we get

$$(3.1) \quad [1-\sqrt{x(1-y)} t]^{-\lambda} {}_2F_1\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; \lambda-\alpha + \frac{1}{2}; \frac{y(1-x)}{\{1+\sqrt{x(1-y)} t\}^2}\right] \\ = \sum_{r=0}^{\infty} \frac{(\lambda)_r}{(\alpha)_r} \left[\frac{1}{4}x(1-y)\right]^{r/2} {}_2F_1\left(\frac{1}{2}\lambda + \frac{1}{2}r, \frac{1}{2}\lambda + \frac{1}{2}r + \frac{1}{2}; 1+\alpha+r; x\right) \\ \times {}_2F_1\left(\frac{1}{2}\lambda + \frac{1}{2}r, \frac{1}{2}\lambda + \frac{1}{2}r + \frac{1}{2}; \lambda-\alpha + \frac{1}{2}; y\right) C_r^{\alpha}(t),$$

which, on replacing  $\lambda$ ,  $\alpha$ ,  $x$  and  $y$  by  $-n$ ,  $\alpha - \frac{1}{2}$ ,  $\frac{x^2 - 1}{x^2}$  and  $\frac{1}{y^2}$ , respectively, and then applying the definition [2, 280]

$$\begin{aligned} C_n^\alpha(x) &= \frac{(\alpha)_n (2x)^n}{n!} {}_2F_1\left(-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; 1-\alpha-n; \frac{1}{x^2}\right) \\ &= \frac{(2\alpha)_n x^n}{n!} {}_2F_1\left(-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \alpha + \frac{1}{2}; \frac{x^2 - 1}{x^2}\right), \end{aligned}$$

is reduced to

$$\begin{aligned} (3.2) \quad &C_n^\alpha[xy - (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}t] \\ &= \frac{\Gamma(2\alpha - 1)}{[\Gamma(\alpha)]^2} \sum_{r=0}^{\infty} (-1)^r \frac{4^r \Gamma(n-r+1) \{\Gamma(\alpha+r)\}^2 (2\alpha+2r-1)}{\Gamma(n+2\alpha+r)} \\ &\quad (x^2 - 1)^{\frac{1}{2}r} (y^2 - 1)^{\frac{1}{2}r} C_{n-r}^{\alpha+r}(x) C_{n-r}^{\alpha+r\frac{1}{2}}(y) C_r^{\alpha-\frac{1}{2}}(t). \end{aligned}$$

(3.2) is due to Gegenbauer [1].

#### R E F E R E N C E S

- [1] L. Gegenbauer, *Wiener Sitzungsberichte*, C. II (1883), p. 942.
- [2] Rainville, E. D. *Special Functions*, New York (1960).

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