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# ON THE TOPOLOGICAL COMPLEMENTATION PROBLEM

*R. Dacić*

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1. *Definition*. Let $X$ be a fixed point set and let $\mathcal{G}$ be a topology on $X$. A topology $\mathcal{G}'$ on $X$ is said to be a complement for $\mathcal{G}$ if and only if the sup topology of $\mathcal{G}$ and $\mathcal{G}'$ is the discrete topology and the inf topology is then trivial topology.

This definition is given in [1] and the problem of complementation is treated. This problem is now solved (see [3]). In [1] the following question is raised:

In the lattice of topologies on an infinite point set $X$, does every topology, which is neither discrete nor trivial, have at least two complements?

Hartmanis (see references of [1]) has stated that if $X$ is a finite set with three or more elements then any topology of $X$, which is neither discrete nor trivial, has more than one complement.

M. P. Berri has constructed a topology on an infinite set $X$ which has more than one complement. That topology is

$$\mathcal{G} = \{A \subset X \mid X \setminus A \text{ is finite}\} \cup \{\Phi\}.$$

In this note we give some other examples of non-uniquely complemented topologies.

A topology $\mathcal{G}$ on $X$ is an *ultraspace* if the only topology on $X$ strictly finer than $\mathcal{G}$ is the discrete topology.

A topology $\mathcal{G}$ on $X$ is an *infraspace* if the only topology on $X$ strictly coarser than $\mathcal{G}$ is trivial topology.

For $p \in X$ and a filter $\mathcal{F}$ on $X$, Frönlich [2] defined $\Sigma(p, \mathcal{F})$ to be the family of sets $P(X - \{p\}) \cup \mathcal{F}$, where $P(X - \{p\})$ is the collection of all subsets of $X$ which do not contain $p$. $\Sigma(p, \mathcal{F})$ is a topology on $X$, and all the sets $\{x\}$ $(x \neq p)$ are open. Frönlich proved that ultraspaces on $X$ are exactly the topologies of the form $\Sigma(x, \mathcal{U})$ where $x \in X$ and $\mathcal{U}$ is an ultrafilter on $X$, $\mathcal{U} \neq \mathcal{U}(x)$. We shall now construct complements for ultraspaces.

Theorem 1. *Every ultraspace topology on $X$ has more than one complement.*

**Proof.** For an ultraspace $\Sigma(x, \mathcal{U})$ the complement is an infraspace $(x, X, \Phi)$ what is evident. We shall now define another topology on $X$. Let $p \in X$, and $A$ be a subset of $X$, $A \overline{\in} \mathcal{U}$. Form a topology on $X$ such that $\{p\}$

be a dense subset of this topology. If $\{B_m\}$ ($m \in M$ — an index set) is a partition of a $A$, then the base for that topology is consisting of the sets $B_m \cup \{p\}$, $X$, and $\{p\}$. Call that topology $p$ — topology, and denote it by $\mathcal{T}_p(A)$. There is no set in $\Sigma(p, \mathcal{U})$ which is a member of $\mathcal{T}_p(A)$ except $X$ and $\varnothing$. So inf $(\Sigma(p, \mathcal{U}), \mathcal{T}_p(A))$ is trivial topology. Since $\{p\}$ is open in $\mathcal{T}_p(A)$ all the sets of the form $\{x\}$, $x \in X$, are open in sup $(\Sigma(p, \mathcal{U}), \mathcal{T}_p(A))$ and so sup is discrete. Choosing different subsets $A$ of $X$ we obtain different complements of $\Sigma(p, \mathcal{U})$. The theorem is proved.

*Remark.* The theorem 1. is valid if ultraspace is replaced by $\Sigma(p, \mathcal{F})$ where $\mathcal{F}$ is not ultrafilter.

Theorem 2. *Every infraspace is non-uniquely complemented.*

**Proof.** Let $\mathcal{T}' = \{X, A \subset X, \Phi\}$ be an infraspace (every infraspace is of that form). Define a topology $\mathcal{T}$ on $X$ in the following way:

$$\mathcal{T} = P(X-A) \cup \mathcal{T}_p(A) \qquad (p \in A)$$

with $B_m = \{m\}$, $m \in A$. It is easily seen that $\mathcal{T}$ is really a topology, in which $A$ is not open.

Therefore inf $(\mathcal{T}', \mathcal{T})$ is a trivial topology. For $x \in X-A$ we have $\{x\} \in \mathcal{T}$ and consequently $\{x\} \in \sup(\mathcal{T}', \mathcal{T})$. For $x \in A$ we have $\{x\} = \{x, p\} \cap A$ so that $\{x\} \in \sup(\mathcal{T}', \mathcal{T})$ for all $x \in X$. One complement of $\mathcal{T}'$ is constructed. To obtain other it is enough to take a point $q \in X-A$, $q \neq p$, and form a topology in the above manner. The theorem is proved.

*Remark.* In these theorems $X$ is supposed to be infinite set. The non--uniqueness of the complements of topologies on a finite set is known (see [1]).

Theorem 3. *If a topological space $X$ is a direct sum of an ultraspace $X'$ and an arbitrary topological space $X''$ then $X$ has non-unique complement.*

**Proof.** We have proved that the space $X'$ has non-unique complement. According to [3], theorem 7, the topology $\mathcal{T}/X''$ ($\mathcal{T}$ is considered topology of $X$ and $\mathcal{T}/X''$, the topology of subspace $X''$) has complement. Denote it by $\mathcal{T}'/X''$. Consider two different complements $\mathcal{T}_1'$ and $\mathcal{T}_1''$ of the topology $\mathcal{T}_1$ of the space $\Sigma(p, \mathcal{U})$. Then the direct sum topologies $\mathcal{T}_1 + \mathcal{T}'/X''$ are different topologies on $X$ which are complements of $\mathcal{T}$. The theorem is proved.

Theorem 4. *If the space $(X, \mathcal{T})$ is direct sum of an interspace $(X', A \subset X, \Phi)$ and a topological space $(X'', \mathcal{T}/X'')$ then $\mathcal{T}$ is non-uniquely complemented.*

In the proof we use the theorem 2. and the same procedure as in the theorem 3.

Theorem 5. Let topology $\mathcal{T}$ *of $X$ is consisted of the disjoint sets $A_i$ $(A_i \neq X)$ and $X \setminus \bigcup_i A_i$ is consisted of more than one point, then $\mathcal{T}$ is non-uniquely complemented.*

**Proof.** Denote by $S = \bigcup_i A_i$ and consider $\mathcal{T}_p(S)$, for $m \in S$ and $p \in X \setminus S$. Then $X \setminus S$ has discrete topology. We shall show that topology $\mathcal{T}'$ which base is $\mathcal{T}_p(S) \cup P(X \setminus S)$ is complement of $\mathcal{T}$. We shall first show that $\sup(\mathcal{T}, \mathcal{T}')$ = discrete topology. Let $x \in X \setminus S$ then $\{x\} \in \mathcal{T}'$ and $\{x\} = \{x\} \cap X$ $(X \in \mathcal{T})$

and consequently $\{x\}$ is open in sup $(\mathcal{G}, \mathcal{G}')$. For $x \in S$ we have $0 \in \mathcal{G}'$, which has $\{p\}$ as dense subset and $0 \cap A_i = \{x\}$ for some $A_i$. So sup $(\mathcal{G}, \mathcal{G}')$ is discrete topology. Show next that inf $(\mathcal{G}, \mathcal{G}')$ = trivial topology. Let $0 \in$ inf $(\mathcal{G}, \mathcal{G}')$. Then 0 is not of the form $\bigcup_{i \in J'} A_i$ for any $J' \subset J$ that is $0 \supset S$ since $\bigcup_{i \in J'} A_i$ as element of $\mathcal{G}$ has not $\{p\}$ as a dense subset. On the other hand, if $0 \cap S \cap (X \setminus S) \neq \varnothing$ it follows that $X \setminus S \subset 0$ and so $0 = X$.

2. Theorem 6. *Every $T_1$-topology, which has no isolated points is non-uniquely complemented.*

**Proof.** This theorem needs the following lemma.

Lemma 1. *If sup $(\mathcal{G}, \mathcal{G}') = P(X)$ = discrete topology, then sup $(\mathcal{G}, \mathcal{G}'') = P(X)$ for all $\mathcal{G}'' \supset \mathcal{G}'$.*

The proof of this lemma is obvious.

**Proof of theorem.**

Let $\mathcal{G}$ be any given $T_1$-topology on $X$ which has no isolated points, and $\mathcal{G}'$ its complement which exists according to the theorem 7.7 of [3]. Let $x \in X$, $\{x\} \bar{\in} \mathcal{G}'$. Put $\mathcal{G}'' = T' \cup \{x\}$. We obtain a topology $\mathcal{G}''$ in which $x$ is isolated. We shall prove that $\mathcal{G}''$ is also a complement of $\mathcal{G}$. According to the lemma, it is enough to prove that inf $(\mathcal{G}, \mathcal{G}'') = \{\varnothing, X\}$. Since $\mathcal{G}$ has no isolated point the set $\{x\}$ is not open in $\mathcal{G}$. Suppose $G \in \mathcal{G} \cap \mathcal{G}''$. If $x \bar{\in} G$, then $G = \varnothing$ according to the facts that open sets in $X \setminus \{x\}$ are the same, both in $\mathcal{G}'$ and $\mathcal{G}''$, and that $\mathcal{G}'$ is the complement of $\mathcal{G}$. Consequently the common open sets in $\mathcal{G}$ and $\mathcal{G}''$ must contain $x$. Make two representations for $G$: $G = \{x\} \cup 0$ $(0 \in \mathcal{G}')$ and $G = \{x\} \cup 0'$ $(0' \in \mathcal{G})$. If $G \neq X$, we shall prove that the latter representation is impossible even if $x \in 0'$, and thus the desired proof will be completed. First of all it is impossible $x \bar{\in} 0 \cap 0'$, for otherwise $0 = 0'$, which is contrary to the hypothesis that inf $(\mathcal{G}, \mathcal{G}') = \{\varnothing, X\}$. The same is obtained by supposition $x \in 0 \cap 0'$. The supposition $x \in 0$ and $x \bar{\in} 0'$ leads to the conclusion that $x$ is an isolated point in $\mathcal{G}$ which is contrary to the hypothesis of the theorem. The only possibility is then $x \bar{\in} 0$ and $x \in 0'$. Then $0 = 0' \setminus \{x\}$. Since $\mathcal{G}$ is $T_1$-topology (and $\{x\}$ is not open) we have $0' \setminus \{x\}$ $(= 0)$ is open in $\mathcal{G}$ and inf $(\mathcal{G}, \mathcal{G}')$ is not trivial, contrary to the hypothesis that $\mathcal{G}'$ is the complement of $\mathcal{G}$. Consequently there is no $G \neq X$ open in $\mathcal{G}''$ and $\mathcal{G}$. Hence $\mathcal{G}''$ is complement of $\mathcal{G}$. Since $\mathcal{G}'' \neq \mathcal{G}'$, the theorem is proved.

*On the S-property.*

Let $(X, \mathcal{G})$ be a topological space which has isolated points. Denote this set of isolated points by $I_1$. Let us consider the subspace $\mathcal{G} \mid (X - I_1)$ and denote by $I_2$ the set of its isolated points.

Suppose that $\alpha$ is an ordinal number and that the set $I_\alpha$ has been defined for all $\alpha < \beta$. We define $I_\beta$ to be the set of isolated points of $\mathcal{G} \mid (X - \bigcup_{\alpha < \beta} I_\alpha)$.

If $|X| < |\gamma|$, $|X|$ means the cardinality of $X$, the family of disjoint sets $\{I_\alpha \mid \alpha < \gamma\}$ is inductively defined.

Suppose that there exists some $\alpha_0$ such that $I_{\alpha_0 + 1} = \varnothing$ and $I_{\alpha_0} \neq \varnothing$. Suppose further that $X - \bigcup_{\alpha \leqslant \alpha_0} I_\alpha \neq \varnothing$, and that $\mathcal{G} \mid (X - \bigcup_{\alpha \leqslant \alpha_0} I_\alpha)$ is $T_1$—topology. If a space $X$ has this property we say that $X$ has the property $(S)$.

Theorem 7. *If a topological space $(X, \mathcal{G})$ has the property $(S)$, the topology $\mathcal{G}$ is non-uniquely complemented.*

**Proof.** First of all we shall point out that for $p \in X_1 = X - \bigcup_{\alpha \leqslant \alpha_0} I_\alpha$, there is no open set $G$ of $\mathcal{G}$ such that $\{p\} = X_1 \cap G$. Really if such a set $G$ exists the point $p$ should be an isolated point of $X_1$ contrary to the definition of $X_1$.

Since $\mathcal{G}_1 = \mathcal{G} \mid X_1$ is a $T_1$—topology which has no isolated point, it has (see [3], theorem 7.7) a complement $\mathcal{G}_1'$. Take a point $p \in X_1$ and denote by $A$ the set $\bigcup_{\alpha \leqslant \alpha_0} I_\alpha$. On the set $\{p\} \cup A$ define a topology of the form $\mathcal{G}_p(A)$ with $B_\alpha = I_\alpha$ and a property that if an open set of $\mathcal{G}_p(A)$ contains $I_\alpha$, it must contain $I_\beta$ for $\beta > \alpha$ (see the proof of the theorem 1.). The topology $\mathcal{G}'$ on $X$ produced by the sets $\mathcal{G}_1' \cup \mathcal{G}_p(A)$ is a complement of the topology $\mathcal{G}$.

Since no one of open sets of the form $\{p\} \cup 0$ with $0 \subset \bigcup_{\alpha \leqslant \alpha_0} I_\alpha$ belongs to $\mathcal{G}$ (or otherwise $\{p\}$ should be isolated point of $X_1$, which is contrary to the hypothesis that $X_1$ has no isolated points) no one of open sets of $\mathcal{G}_p(A)$ belongs to $\mathcal{G}$. Consider an open set $O$ of $\mathcal{G}' \cap \mathcal{G}$. Since $O \in \mathcal{G}'$ it, can ce represented in the form $O_1 \cup O_2$, $O_1 \in \mathcal{G}_1'$ and $O_2 \in \mathcal{G}_p(A)$. According to the proof of theorem 6. $O_1 \supset X_1$. If $M$ denotes a subset of ordinal numbers $\alpha < \alpha_0$, we have $O = X_1 \cup (\bigcup_{\beta \in M} I_\beta)$. Let $\beta_0$ be the least element of $M$. Since for all $x \in I_\beta$ there exists an open set $G$ in $\mathcal{G}$ contained in $\bigcup_{\alpha \leqslant \beta_0} I_\alpha$ and $G \cap I_{\beta_0} = \{x\}$. We have that the set $G = \bigcup_{x \in I\beta_0} G_x \subset \bigcup_{\alpha \leqslant \beta_0} I_\alpha$ is an element of $\mathcal{G}$. From the hypothesis $O \in \mathcal{G}$ it follows that $I_{\beta_0}$ is open in $\mathcal{G}$ and hence $\beta_0 = 1$, that is $0 = X$. So $\inf(\mathcal{G}, \mathcal{G}')$ is $(X, \varnothing)$.

Prove next that sup $(\mathcal{G}, \mathcal{G}')$ is discrete topology. If $x \in X_1$ this assertion follows from the fact that $\mathcal{G}' \mid X_1$ is finer than $\mathcal{G}_1'$, and $\mathcal{G}_1'$ is a complement of $\mathcal{G} \mid X_1$. Take, therefore, $x \in X - X_1$. Let $x \in I_{\alpha_0}$; then there exists $O \in \mathcal{G}$, $O \subset \bigcup_{\alpha \leqslant \alpha_0} I_\alpha$ and $O \cap I_\alpha = \{x\}$. But the set $\{p\} \cup \bigcup_{\beta \geqslant \beta_0} I_\beta$ is open in $\mathcal{G}'$ and consequently $\{x\} \in \sup(\mathcal{G}, \mathcal{G}')$, that is sup $(\mathcal{G}, \mathcal{G}')$ is a discrete topology, what was to be proved.

To prove the theorem, take another point $p_1 \in X_1$, $p_1 \neq p$. The existence of an infinite number of such points $p$ is guaranteed by the following reason. If $X_1$ has a finite number of points $p_1, \ldots, p_n$ and since $\mathcal{G}_1$ is $T_1$—topology, the space $(X_1, \mathcal{G}_1)$ must be discrete and, then all $p_i\,(i = 1, \ldots, n)$ are isolated points, contrary to the hypothesis of the theorem.

Theorem 8. *If $X = \bigcup_{\alpha \leqslant \alpha_0} I_\alpha$, then $\mathcal{G}$ is non-uniquely complemented.*

**Proof.** $\mathcal{G}'$ is defined in the following way: all $O \in \mathcal{G}$ contain $I_{\alpha_0}$; if $x \in I_\alpha$, $x \in O$, then $I_\alpha \subset O$, and if $O$ contains $I_\alpha$, then all $I_\beta$, for $\beta < \alpha$, are contained in $O$. It is easily seen that $\mathcal{G}'$ defined in this way, is really a topology on $X$. By the arguments used in the proof of the theorem 7. we are convinced that $\mathcal{G}'$ is really a complement of $\mathcal{G}$. To obtain another complement $\mathcal{G}''$ take a point $p \in I_{\alpha_0}$ and make that point isolated in $\mathcal{G}'$. First of all we must prove that $I_{\alpha_0} \neq \{p\}$. If $\{p\} = I_{\alpha_0}$ then $p$ is an isolated point of $I_{\alpha_0 - 1}$ contrary to the hypothesis that $\alpha_0$ is the least ordinal number $\alpha$ for which $I_{\alpha_0} \neq \varnothing$. So $I_{\alpha_0}$ is consisted of at least two points. We shall next prove that $\mathcal{G}''$ is really a complement of $\mathcal{G}$. According to the lemma it is enough to prove that inf $(\mathcal{G}, \mathcal{G}'') = (X, \varnothing)$. To obtain that, we must prove that $I_{\alpha_0}$ is not open in $\mathcal{G}$. Suppose contrary. As in the proof of theorem 7, we have that $\bigcup_{\alpha < \alpha_0} I_\alpha$ is open in $\mathcal{G}$.

Since $x \in I_{\alpha_0}$ is an isolated point of $\mathcal{T} \,|\, (X - \bigcup\limits_{\alpha<\alpha_0} I_\alpha)$ there exists an open set $O$ in $\mathcal{T}$ such that $\{x\} = O \cap I_\alpha$ and being intersection of two open sets in $\mathcal{T}$, $\{x\}$ is open in $\mathcal{T}$, which leads to the conclusion that $\mathcal{T}$ is a discrete topology which is contrary to the hypothesis that all considered topologies are neither discrete nor trivial. So $I_{\alpha_0}$, as well as $\bigcup\limits_{\alpha>\beta,\ \beta<\alpha_0} I_\alpha$, is not open in $\mathcal{T}$.

Suppose now that there exists a set $O$ open in both topologies $\mathcal{T}$ and $\mathcal{T}''$. Evidently $O \neq \{p\}$. If $p \,\overline{\in}\, O$ then no one point of $I_{\alpha_0}$ is in $O$ (according to the definition of $\mathcal{T}'$ and consequently $\mathcal{T}''$) and $O = \varnothing$. So $p \in O$. We have already proved that $I_{\alpha_0} \overline{\in} \mathcal{T}$, even more $\bigcup\limits_{\alpha\geqslant\beta,\ \beta>\alpha_0} I_\alpha \overline{\in} \mathcal{T}$. So $O$ as element of $\mathcal{T}$ must contain a point $x \in I_\alpha, \alpha < \alpha_0$. But as an element of $\mathcal{T}''$ the set $O$ must contain the whole $I_\alpha$ and so $O = X$. Consequently inf $(\mathcal{T}, \mathcal{T}'') = (X, \varnothing)$.

Taking into account theorems 6, 7 and 8 we have the following

T h e o r e m 9. *Every $T_1$-topology on any infinite set is non-uniquely complemented.*

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Institut Mathématique
Beograd