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ON THE TRANSFORMATION OF THERMAL BOUNDARY-LAYER EQUATIONS

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Abstract. This paper deals with a transformation of the thermal boundary-layer equation into a universal form in the sense of the paper [1]. The question of solving such an equation is treated only principally, as it is easy to obtain it by the procedure given in the paper [3].

Introduction. In the paper [1] we mentioned some weaknesses of the method [2] in comparison to [1]. Here, we mention that the same weaknesses hold for the method [3], as both belong to the same class of solutions. Therefore, our task is to improve the method [3] in the same way in which the method [1] improves the method [2]. As in the paper [1] we considered that question in details, here, we consider only the transformation of the basic equation, and then point out to the way for solving obtained universal equation. For this purpose we first give a brief review of the paper [1]. Namely, in that paper we have assumed that the stream function ψ has the form

$$\psi(x, y, t) = A^{-1} U(x, t)\, \delta_p^*(t)\, \mathfrak{F}(\eta; d), \tag{1}$$

where $\mathfrak{F}(\eta; d) = \mathfrak{F}(\eta;\ d_1, \ldots, d_n)$ together with its derivatives are real continuous functions defined on the product space $E^1 \times W$, where W is an arbitrary fixed set in Euclidean n-space E^n, and d is an essentially bounded function from $E^1 \times I$, $I = [0, \infty)$, to W with continuous first derivatives. Further, we showed that the function $d(x, t)$ could be factorized as follows

$$d(x, t) = f(x)\, h(t) + g(t), \tag{2}$$

where g, h and f were given to satisfy certain types of ordinary differential equations [1]. By using (1) and differential equations for given parameters, the basic equation of the velocity boundary layer is transformed into a universal form. Furthermore, we gave the solution of that equation and in details studied, so named „simple solutions“ which corresponded to one-parameter approximation of the function $\mathfrak{F}$, namely for the case $d = d_1$. Also, it is shown that the one-parameter approximation, in particular cases, gives the same values of boundary-layer magnitudes as those which can be found in [4]. As the temperature field can be obtained only after the velocity field (u, v) then we have a stage for solving such a problem.

The basic problem. We consider, as in [3], the equation of the temperature distribution for the case of two-dimensional forced-convection flows. This equation is

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{gC_p}\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\right)(U-u)+\frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{gC_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

with boundary conditions

$$T=T_w(x,t)\ or\ \frac{\partial T}{\partial y}=0,\ y=0,$$

$$T=T_\infty, \qquad y=\infty, \tag{4}$$

where T is a real valued function of the class C^k $(k=0,1,\ldots)$ in $E^2\times I$. Now, we assume that the function T_w has the same form as in [3], namely

$$T_w(x,t)-T_\infty=S(x)\,\theta(t), \tag{5}$$

with S and θ real valued functions in E^1 and I respectively, and of the class C^k. In order to achieve the purpose mentioned in the introduction, we consider two sets A and B in Euclidean n-space. Respective elements of these sets, $a_k\in A$ and $b_k\in B$, are assumed as follows

$$a_k=\frac{1}{\theta}\frac{d^k\theta}{dt^k}z^{*k} \qquad b_k=\frac{V^k}{S}\frac{d^kS}{dx^k}, \tag{6}$$

where $z^*=\frac{1}{\nu}\delta_p^{*2}$ and δ_p^* is the displacement thickness. It is easy to verify that mentioned elements satisfy the following ordinary differential equations

$$\frac{\theta}{\theta'}a_1a_k'=(-a_1+2kF)\,a_k+a_{k+1}, \qquad Vb_k'=(-b_1+kf_1)\,b_k+b_{k+1}. \tag{7}$$

Next we assume that the equation (3) may be taken in the form

$$T(x,y,t)=$$

$$=T_\infty+S(x)\,\theta(t)\left[\mathcal{H}(\eta;f,g,h,a,b;\sigma)+\frac{1}{gC_p}\frac{V^2(x)\Omega^2(t)}{S(x)\theta(t)}\mathcal{K}(\eta;f,g,h,b;\sigma)\right], \tag{8}$$

where $\mathcal{H}(\eta;f,\ldots)=\mathcal{H}(\eta;f_1,\ldots f_k,\ldots)$ is a continuous function which represents the solution of a heating, respectively cooling problem, and $\mathcal{K}(\eta;f,\ldots)=\mathcal{K}(\eta;f_1,\ldots.f_k,\ldots)$ is also a continuous function which represents the solution of the thermometer problem.Taking into account (8) and expressions (6) and (7), the differential equation (3) is separated in two partial equation of universal forms, in the sense of [1],

$$P(\mathcal{H})=\frac{1}{A^2}h_1\left\{\sum_{k=1}^{n}[\mathfrak{F}_\eta(\mathcal{H}_{f_k}\mu_k+\mathcal{H}_{b_k}\beta_k)-\mathcal{H}_\eta\mathfrak{F}_{f_k}\mu_k]+b_1\mathfrak{F}_\eta\mathcal{H}-f_1\mathfrak{F}\mathcal{H}_\eta\right\}, \tag{9}$$

respectively

$$\overline{P}(\mathcal{K})=\frac{1}{A^2}h_1\left\{(b_1+2f_1)\mathfrak{F}_\eta\mathcal{K}-f_1\mathfrak{F}\mathcal{K}_\eta+\sum_{k=1}^{n}[\mathfrak{F}_\eta(\mathcal{K}_{f_k}\mu_k+\mathcal{K}_{b_k}\beta_k]-\mathcal{K}_\eta\mathfrak{F}_{fk}\mu_k]\right\}$$

$$-\left[\frac{1}{A^2}(1-\mathfrak{F}_\eta)(g_1+f_1h_1)+\mathfrak{F}_{\eta\eta}^2\right], \tag{10}$$

with boundary conditions

$$\mathcal{H}=1, \qquad \mathcal{K}_\eta=0, \qquad \eta=0,$$

$$\lim_{\eta\to\infty}\mathcal{H}=\lim_{\eta\to\infty}\mathcal{K}=0. \tag{11}$$

These equations are, so named, universal equations, in the sense that neither equations nor boundary conditions depend on the particular problem data. What are preferences of these equations in comparison to those in [3] can be concluded from [1]. In obtained equations two operators P and $\overline{P}$ appear. Forms of these operators are;

$$P=\frac{1}{\sigma}D_\eta{}^2+\frac{F}{A^2}\eta\,D_\eta-\frac{a_1}{A^2}-\frac{1}{A^2}\sum_{k=1}^{n}(\theta_k D_{h_k}+\lambda_k D_{g_k}+\alpha_k D_{a_k}), \tag{12}$$

where $D_m=\dfrac{\partial}{\partial_m}$, and the operator $\overline{P}$ has the same form, only instead of a_1 in $\overline{P}$ ought to stand $2g_1$, and the term $-\dfrac{1}{A^2}\sum_{k=1}^{n}\alpha_k D_{a_k}$ does not exist in $\overline{P}$.

Thus, the basic goal of this paper is achieved. Now, we shall point out how to solve obtained partial equations. They are very simple to be solved if we follow procedures given in [3] and [1]. So it is obvious from [3] that suitable forms of the functions $\mathcal{H}(\eta; f, g, h, a, b; \sigma)$ and $\mathcal{K}(\eta; f, g, h, b; \sigma)$ are:

$$\mathcal{H}=\mathcal{H}_0+f_1\,\mathcal{H}_1+b_1\,\mathcal{H}_{1a}+\cdots,$$

$$\mathcal{K}=\mathcal{K}_0+f_1\,\mathcal{K}_1+b_1\,\mathcal{K}_{1a}+\cdots. \tag{13}$$

In that manner partial equations (9) and (10) would be reduced to systems of partial equations. Next, we could introduce linear combinations by which one could make decompositions of given functions $\mathcal{H}_0, \ldots$ and $\mathcal{K}_0, \ldots$. These linear combinations could be carried out in a simple way, as in [3] and [1]. By introducing such linear combinations, systems of partial equations would be reduced to systems of ordinary differential equations. The general solution of such a type of equations is given in [3]. We omit details about these questions.

Simple solutions. In the paper [1] is pointed out that simple solutions are those with one-parameter approximations and assumed linearities. Here, we shall slightly discuss these solutions. For them, functions $\mathcal{H}$ and $\mathcal{K}$ given by (13) are reduced to the forms

$$\mathcal{H}=\mathcal{H}_0+f_1\,\mathcal{H}_1+b_1\,\mathcal{H}_{1a}, \quad \mathcal{K}=\mathcal{K}_0+f_1\,\mathcal{K}_1+b_1\,\mathcal{K}_{1a}. \tag{14}$$

Subsitituting these forms of functions $\mathcal{H}$ and $\mathcal{K}$ into (9) and (10) we obtain systems of partial equations

$$P(\mathcal{H}_0)=0, \quad P(\mathcal{H}_1)=-h_1A^{-2}\mathfrak{F}_0\,\mathcal{H}_{0\eta}, \quad P(\mathcal{H}_{1a})=h_1A^{-2}\mathfrak{F}_{0\eta}\,\mathcal{H}_0,$$

$$\overline{P}(\mathcal{K}_0)=-g_1A^{-2}(1-\mathfrak{F}_{0\eta})-\mathfrak{F}_{0\eta\eta}^{2}, \quad \overline{P}(\mathcal{K}_1)=h_1A^{-2}[(2\,\mathfrak{F}_{0\eta}\,\mathcal{K}_0-\mathfrak{F}_0\,\mathcal{K}_{0\eta})-$$

$$-(1-\mathfrak{F}_{0\eta})]+g_1\,\mathfrak{F}_{1\eta}-2\mathfrak{F}_{0\eta\eta}\,\mathfrak{F}_{1\eta\eta}, \quad \overline{P}(\mathcal{K}_{1a})=h_1A^{-2}\mathfrak{F}_{0\eta}\,\mathcal{K}_0 \tag{15}$$

Furthermore, by linear combinations we can conclude that functions which appear in (14) may be decomposed as follows

$$\mathcal{H}_0=\mathcal{H}_{00}+g_1\,\mathcal{H}_{01}+b_{11}\mathcal{H}_0, \quad \mathcal{H}_1=h_1\,\mathcal{H}_{11}, \quad \mathcal{H}_{1a}=h_1\,\mathcal{H}_{11a},$$

$$\mathcal{K}_0=\mathcal{K}_{00}+g_1\,\mathcal{K}_{01}, \quad \mathcal{K}_1=h_1\,\mathcal{K}_{11}, \quad \mathcal{K}_{1a}=h_1\,\mathcal{K}_{11a} \tag{16}$$

and accordingly systems of partial equations (15) are to be reduced to systems of ordinary differential equations for determination of unknown functions $\mathcal{H}_{00}, \ldots$ and $\mathcal{K}_{00}, \ldots$. These systems we shall not write as it is a simple matter to obtain them. We point only out that in [3] we considered such a type of equations and gave its general solution. Moreover, in [1] is shown that simple solutions give satisfying results for practical considerations. Their very simple forms do that we can arrive at a final result very quickly.

It remains still to point out that one could carry out all formulas for boundary-layer magnitudes as in [3]. Also, notions of the proper temperature and temperature criterion could be introduced. Details about all these questions we omit considering them very simple.

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