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# ON THE UNIVERSAL FORM OF UNSTEADY INCOMPRESSIBLE BOUNDARY-LAYER EQUATION AND ITS SOLUTION

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*Abstract.* This paper deals with the question of universality of the basic equation of unsteady incompressible laminar boundary layers in the sense that neither equation nor boundary conditions depend on particular problem conditions and with question of its solving. The universality is achieved by assuming that $U(x, t)=V(x)\,\Omega(t)$, where $V(x)$ and $\Omega(t)$ are sufficiently smooth functions, and transferring sets of parameters, here introduced, which express the influence of external conditions characteristic for each particular problem into the number of independent variables. The solution of the obtained universal equation is given in the form of series expansion in mentioned parameters.

**1.** *Introduction.* To arrive at the object of this paper we shall shortly stay on the analysis of some methods for solving steady boundary layers. For this purose we choose, on one side, a representative of former accurate methods, the well-known method by Görtler [4], and on the other side the new method by Loitsianskii [6]. In both mentioned methods solutions are given in the form of series expansion in certain parameters. So the velocity profile by Görtler is given in he form

$$\frac{u}{V}=\mathcal{G}(\xi,\ \eta;\ \{\beta_k\}), \tag{1.1}$$

where $\mathcal{G}(\xi, \eta; \{\beta_k\})$ is a function represented by means of series expansion in $\xi$ and $\beta_k$; $\xi$ and $\eta$ are independent variables given as

$$\xi=\frac{1}{\nu}\int_0^x V(x)\,dx, \quad \eta=\frac{V(x)}{\nu\sqrt{2\xi}}\,y, \tag{1.2}$$

while the set of parameters $\{\beta_k\}$, where $\beta_k$ are constants, is obtained as coefficients at $\xi$ in expanding the principal function

$$\beta(\xi)=\frac{V'\,\nu\,2\xi}{V^2}, \quad x=x(\xi), \tag{1.3}$$

in power series. Thus, in Görtler's method parameters $\beta_k$ only characterise the external distribution.

The Loitsianskii's profile, however, is obtained as follows

$$\frac{u}{V}=\mathscr{L}(\eta^*;\{f_k\}), \tag{1.4}$$

where $\mathscr{L}(\eta^*;\{f_k\}$ is a function in $\eta^*$ and a functional in $\{f_k\}$ given in the form of series expansion in $f_k$; $\eta^*$ is the independent variable given as

$$\eta^*=B_0\frac{y}{\delta^{**}}, \tag{1.5}$$

where $\delta^{**}$ is the momentum thickness, and $B_0$ a constant; $\{f_k\}$ is a set of parameters which are, in this case, functions, namely $f_k=f_k(x)$, and they are given by

$$f_k=V^{k-1}\frac{d^k V}{dx^k}(z^{**})^k, \tag{1.6}$$

where $z^{**}=\frac{\delta^{**2}}{\nu}$. Through these parameters we have the entry of the real boundary layer development.

Let us now analyse the mentioned parameters and the obtained solutions. The principal function $\beta$ essentially corresponds to the first parameter $f_1$ in the set of parameters $\{f_k\}$, considering that it can be shown that $\delta^*$ as a measure of boundary layer is proportional to $\frac{\nu\sqrt{2\xi}}{V}$, hence $\beta$ is proportional to $V'z^*$, where $z^*=\frac{\delta^{*2}}{\nu}$. Thus, we have essentially an agreement of parameters $f_1$ and $\beta$, although the real expressions for both parameters are different. For the case of an one-parameter solution Loitsianskii obtained

$$f_1=a\frac{V'}{V^b}\int_0^x V^{b-1}\,dx,$$

where $a=0{,}4408$ and $b=5{,}714$, and Görtler got

$$\beta=2\frac{V'}{V^2}\int_0^x V\,dx.$$

The difference is obvious, and that is normal, because Görtler's parameter $\beta$ is not obtained as a real corollary of the boundary layer development, but it expresses the assumed influence of external conditions by means of the assumption of the form of independent variables.

Furthermore, the solution (1.1), in certain manner, represents an one--parameter solution

$$\frac{u}{V}=\mathcal{G}^{*}(\eta;\ \beta). \tag{1.7}$$

But, in Görtler's method, the mentioned functional dependence is realized in the form of series expansion in $\xi$ and parameters $\beta_k$, and in the method by Loitsianskii it is already realized by the linear term of the expansion in parameters $f_k$. Thus, what is by Görtler achieved by means of a sequence of terms in the mentioned series expansion that is by Loitsianskii achieved by means of the linear term and with the solution in the form of an integral. Moreover, by Loitsianskii, parameters of high order appear, which in certain manner indicate the influence of the curvature of the external distribution, thus also of the contour, which does not exist in Görtler's method. Thus, Görtler's method may give good results only on contours with sharp front edge, namely, which are close to the flat plate, while the method by Loitsianskii, on arbitrary contours with very small number of terms in the series expansion give the results entirely close to the exact ones. Namely, it is shown practically that it is quite sufficient to stay on the second parameter of the given set. We must still mention that in difference to Görtler, where the universality of only equations for determination of coefficients of series expansion in $\xi$ and $\beta_k$ is achieved, by Loitsianskii the universality of the basic equation of boundary layers is already achieved.

All what is said concerning Görtler's and Loitsianskii's methods can also be said for the method [1] and the present one which is given in this paper. Namely, the same weaknesses of the method [4] hold for the method [1], too. In the present method an improvement of the method [1] is achieved in the sense in which the method [6] improves the method [4]. In this case three sets of parameters are introduced, which practically represent subsets of a given set, about which will be detailly said in the paper. Introducing the mentioned parameters as independent variables, the universality of the basic equation of boundary layers is achieved. The solution of this equation is given in the form of series expansion in quoted parameters. Coefficients in the given expansion up to the linearity are obtained in the closed form. Moreover, in the paper detailly are considered so named simple solutions, namely one--parameter solutions with assumed linearity, and two examples are solved. Finally, the question of the mentioned series is touched.

**2.** *Preliminaires and main results.* In this section we first state the problem for consideration. then explain some known results needed for the work in sequel and finally we set the main result of the paper.

Basic equations defining unsteady fluid flows in two-dimensional case by introducing the stream function are reduced to an equation

$$\psi_{yt}+\psi_y\,\psi_{xy}-\psi_x\,\psi_{yy}=U_t+UU_x+\nu\psi_{yyy}, \tag{2.1}$$

with boundary conditions

$$\begin{aligned}
&\psi_y=U(x,\,t),\quad \psi_x=0, && y=0,\ t=0,\\
&\psi=\psi_x=\psi_y=0, && y=0,\ t>0,\\
&\psi_y\to U(x,\,t), && y\to\infty,
\end{aligned} \tag{2.2}$$

where indices denote partial derivatives with respect to corresponding coordinates, and the remaining values are well-known in the boundary layer theory.

For the later work we assume that the main stream velocity $U(x, t)$ is given in the form $U(x, t) = V(x)\,\Omega(t)$, and that functions $V(x)$ and $\Omega(t)$ are of the class $C^k$, $0 \leqslant k \leqslant \infty$ on considered domains.

Now, we shall occupy ourselves with some known results concerning the question of the flow past a flat plate, considering that we shall have, locally at a spot on the contour, the same problem. For the case of flows with the velocity $\Omega(t)$ the momentum equation, as it is well-known [3], has the form

$$\frac{\partial}{\partial t}(\Omega\,\delta_p^*) = \frac{1}{\rho}\,\tau, \tag{2.3}$$

where $\delta_p^*(t) = \int\limits_0^\infty \left(1 - \frac{u_p}{\Omega}\right) dy$ is the displacement thickness, and $\tau = \mu\,\frac{\Omega}{\delta_p^*}\,\zeta$ is the skin friction. By introducing the local form parameter

$$g_1 = \frac{\Omega'}{\Omega}\,z^* \tag{2.4}$$

where $z^* = \frac{\delta_p^{*2}}{\nu}$ the equation (2.3) is reduced to

$$g_1' + g_1\left(3\frac{\Omega'}{\Omega} - \frac{\Omega''}{\Omega'}\right) = 2\,\frac{\Omega'}{\Omega}\,\zeta. \tag{2.5}$$

From (2.4) we have

$$z^* = g_1\,\frac{\Omega}{\Omega'}. \tag{2.6}$$

Obviously, the last two equations give

$$z^{*\prime} = 2\,F, \quad \text{where } F = \zeta - g_1 \tag{2.7}$$

Let us consider now, in the introduction quoted, sets of parameters $\{f_k\}$, $\{g_k\}$ and $\{h_k\}$ for every $k \in (1, 2, \ldots)$. We assume these parameters as follows

$$f_k = V^{k-1}\,\frac{d^k V}{dx^k}, \quad g_k = \frac{1}{\Omega}\,\frac{d^k\Omega}{dt^k}\,z^{*k}, \quad h_k = \Omega^k\,z^{*k}. \tag{2.8}$$

It is easy to show that these parameters can be obtained from a general set of parameters $\{d_k\}$ such that

$$d_k = \left(\frac{1}{U}\,\frac{\partial^k U}{\partial t^k} + U^{k-1}\,\frac{\partial^k U}{\partial x^k}\right) z^{*k}. \tag{2.9}$$

Namely, assuming that $U(x, t) = V(x)\,\Omega(t)$ and taking that $z^* = z^*(t)$ we obtain

$$d_k = \left(\frac{1}{\Omega}\,\frac{d^k\Omega}{dt^k} + V^{k-1}\,\frac{d^k V}{dx^k}\,\Omega^k\right) z^{*k}.$$

Thus, the bounded function $d_k$ defined on the product space is to be factorized as follows

$$d_k = g_k + f_k\, h_k . \tag{2.10}$$

In comparison of the parameters $f_k$ with those introduced by Loitsianskii we see that ours represent a part of Loitsianskii's parameters. However, there is a reason for that, because we first consider the local boundary layer development, then its development along the contour. In that manner we have done that local parameters have a principal role. Therefore, we have $z^* = z^*(t)$ and $z^*$ is to be connected with $\Omega$ and its time derivatives to make local parameters. The influence of the contour is through the parameters $f_k$.

By differentiation of above given parameters (2.8) with respect to corresponding coordinates and taking into account (2.7) one finds that they satisfy the following differential equations

$$\begin{aligned} V f_k' &= (k-1) f_1 f_k + f_{k+1}, \\ \frac{\Omega}{\Omega'} g_1 g_k' &= (-g_1 + 2kF)\, g_k + g_{k+1}, \\ \frac{\Omega}{\Omega'} g_1 h_k' &= k\,(g_1 + 2F)\, h_k . \end{aligned} \tag{2.11}$$

Now we can state the main result of this paper. *Transferring parameters $f_k$, $g_k$ and $h_k$, given by* (2.8), *into independent variables one can reduce the basic equation of unsteady laminar boundary layers to the universal form.*

**3.** *Transformation of the basic equation and solving the transformed equation.*

As we have already mentioned in the previous section, we introduce parameters (2.8) as independent variables, then assume the stream function $\psi(x, y, t)$ in the form

$$\psi(x, y, t) = A^{-1} U(x, t)\, \delta_p^*(t)\, \mathfrak{F}(\eta; \{f_k\}, \{g_k\}, \{h_k\}), \tag{3.1}$$

where $\mathfrak{F}(\eta; \{f_k\}, \ldots) = (\eta; f_1, \ldots, f_k, \ldots)$ is a real-valued function of the class $C^n$, $\eta = A \dfrac{y}{\delta_p^*}$, and $A$ is a norming constant. By substitution of (3.1) into (2.1) and taking into account (2.11) we obtain the transformed form of the equation (2.1) as follows

$$L(\mathfrak{F}) = -h_1 \left[ f_1 (1 - \mathfrak{F}_\eta^2 + \mathfrak{F}\mathfrak{F}_{\eta\eta}) + \sum_{k=1}^{n} \mu_k (\mathfrak{F}_{f_k} \mathfrak{F}_{\eta\eta} - \mathfrak{F}_\eta \mathfrak{F}_{\eta f_k}) \right], \tag{3.2}$$

with boundary conditions

$$\begin{aligned} &\mathfrak{F}_\eta = 1, && \eta = 0,\ t = 0, \\ &\mathfrak{F} = \mathfrak{F}_\eta = 0, && \eta = 0,\ t > 0, \\ &\mathfrak{F}_\eta \to 1, && \eta \to \infty . \end{aligned} \tag{3.3}$$

In the above equation $L$ is a linear operator with valuable coefficients of the form

$$(3.4)\qquad L=A^2 D_\eta^3+F\eta D_\eta^2+g_1(1-D_\eta)-\sum_{k=1}^{n}\left(\lambda_k\frac{\partial^2}{\partial\eta\,\partial g_k}+\theta_k\frac{\partial^2}{\partial\eta\,\partial h_k}\right).$$

where $D_\eta^m=\dfrac{\partial^m}{\partial\eta^m}$. Expressions $\mu_k$, $\lambda_k$ and $\theta_k$ are right hand-sides of (2.11), namely $\mu_k=\mu_k(\{f_k\})$, $\lambda_k=\lambda_k(\{g_k\})$ and $\theta_k=\theta_k(\{g_k\},\{h_k\})$.

The equation (3.2) is a universal equation of unsteady laminar boundary layers. It does not contain values characteristic for some given particular problem. We seek its solution in the form of series expansion corresponding to the expansion by Loitsianskii [6], namely

$$(3.5)\qquad \mathfrak{F}(\eta;\{f_k\},\{g_k\},\{h_k\})=\mathfrak{F}_0(\eta;\{g_k\})+f_1\mathfrak{F}_1(\eta;\{g_k\},\{h_k\})$$
$$+f_1^2\mathfrak{F}_{11}(\eta;\{g_k\},\{h_k\})+f_2\mathfrak{F}_2(\eta;\{g_k\},\{h_k\})+\cdots,$$

in which the following combinations appear

$$(3.6)\qquad \mathfrak{F}_0\ll 1\gg;\ \mathfrak{F}_1\ll f_1\gg;\ \mathfrak{F}_{11},\ \mathfrak{F}_2\ll f_1^2, f_2\gg;\ \mathfrak{F}_{111},\ \mathfrak{F}_{12}, \mathfrak{F}_3\ll f_1^3, f_1f_2, f_3\gg,\ldots$$

By substitution of (3.5) into (3.2) we obtain a system of partial equations for determining coefficients of the above series

$$(3.7)\qquad \begin{aligned} &L(\mathfrak{F})=0,\\ &L(\mathfrak{F}_1)=-h_1\left(1-\mathfrak{F}_{0\eta}^2+\mathfrak{F}_0\mathfrak{F}_{0\eta\eta}\right),\\ &L(\mathfrak{F}_{11})=-h_1(-2\mathfrak{F}_{0\eta}\mathfrak{F}_{1\eta}+\mathfrak{F}_0\mathfrak{F}_{1\eta\eta}+\mathfrak{F}_1\mathfrak{F}_{0\eta\eta}),\\ &L(\mathfrak{F}_2)=-h_1(\mathfrak{F}_1\mathfrak{F}_{0\eta\eta}-\mathfrak{F}_{0\eta}\mathfrak{F}_{1\eta}),\\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

with boundary conditions

$$(3.8)\qquad \begin{aligned} &\mathfrak{F}_{0\eta}=0,\ \mathfrak{F}_{1\eta}=\mathfrak{F}_{11\eta}=\cdots=0,\ \eta=0,\ t=0,\\ &\mathfrak{F}_0=\mathfrak{F}_{0\eta}=0,\ \mathfrak{F}_1=\mathfrak{F}_{1\eta}=0,\ldots,\ \eta=0,\ t>0,\\ &\mathfrak{F}_{0\eta}\to 1,\ \mathfrak{F}_{1\eta}\to 0,\ldots,\qquad \eta\to\infty. \end{aligned}$$

In the form, the series expansion (3.5) and the above system of equations are the same as those in the paper [1]. For solving the system (3.7) one can see that the following logical combinations appear

$$\mathfrak{F}_0^0\ll 1\gg;\ \mathfrak{F}_0^1\ll g_1\gg;\ \mathfrak{F}_0^{11},\ \mathfrak{F}_0^2\ll g_1^2,\ g_2\gg;\ \mathfrak{F}_0^{111},\ \mathfrak{F}_0^{12},\ \mathfrak{F}_0^3\ll g_1^3,\ g_1g_2,\ g_3\gg,\ldots$$

$$(3.9)\qquad {}^1\mathfrak{F}_1\ll h_1\gg;\ {}^1\mathfrak{F}_1^1,\ {}^2\mathfrak{F}_1\ll h_1g_1,\ h_2\gg;\ {}^1\mathfrak{F}_1^{11},\ {}^1\mathfrak{F}_1^2,\ {}^2\mathfrak{F}_1^1,\ {}^3\mathfrak{F}_1\ll h_1g_1^2,$$
$$h_1g_2,\ h_2g_1,\ h_3\gg,\ldots,$$
$${}^{11}\mathfrak{F}_{11(2)}\ll h_1^2\gg;\ {}^{11}\mathfrak{F}_{11(2)}^1,\ {}^{12}\mathfrak{F}_{11(2)}\ll h_1^2g_1,\ h_1g_2\gg,\ldots$$

Hence we see that respective solutions ought to be sought in the following forms

$$
(3.10)\qquad
\begin{aligned}
&\mathfrak{F}_0(\eta;\,\{g_k\}) = \overset{0}{\mathfrak{F}}_0(\eta) + g_1 \overset{1}{\mathfrak{F}}_0(\eta) + g_1^2 \overset{11}{\mathfrak{F}}_0(\eta) + g_2 \overset{2}{\mathfrak{F}}_0(\eta) + \cdots,\\
&\mathfrak{F}_1(\eta;\,\{g_k\},\,\{h_k\}) = h_1\, {}^1\mathfrak{F}_1(\eta) + h_1 g_1\, {}^1\overset{1}{\mathfrak{F}}_1(\eta) + h_2\, {}^2\mathfrak{F}_1(\eta) + \cdots,\\
&\mathfrak{F}_{11(2)}(\eta;\,\{g_k\},\,\{h_k\}) = h_1^2\, {}^{11}\mathfrak{F}_2(\eta) + \cdots
\end{aligned}
$$

Moreover, to solve the system (3.7) we must have an explicit form of the function $F(\{g_k\})$. A suitable form of the function $F(\{g_k\})$ is the same as that of the function $\mathfrak{F}_0(\eta;\,\{g_k\})$, namely

$$
(3.11)\qquad F(\{g_k\}) = F_0 + g_1 F_1 + g_1^2 F_{11} + g_2 F_2 + \cdots
$$

Substituting, in such a manner assumed solutions in the system (3.7) we obtain systems of ordinary differential equations. Staying on the second parameters, we have the following system of equations

$$
(3.12)\qquad
\begin{aligned}
&k=0\quad L_0^*(\overset{0}{\mathfrak{F}}_0) = 0,\\
&k=1\quad L_1^*(\overset{1}{\mathfrak{F}}_0) = -\left(1 - \overset{0'}{\mathfrak{F}}_0 + \eta F_1 \overset{0''}{\mathfrak{F}}_0\right),\\
&k=2\quad L_2^*(\overset{11}{\mathfrak{F}}_0) = 2F_1 \overset{1'}{\mathfrak{F}}_0 - \eta\left(F_{11} \overset{0''}{\mathfrak{F}}_0 + F_1 \overset{1''}{\mathfrak{F}}_0\right),\\
&\qquad\quad L_0^*(\overset{2}{\mathfrak{F}}_0) = \overset{1'}{\mathfrak{F}}_0 - \eta F_2 \overset{0''}{\mathfrak{F}}_0,\\
&k=1\quad L_1^*({}^1\mathfrak{F}_1) = -\left(1 - (\overset{0'}{\mathfrak{F}}_0)^2 + \overset{0}{\mathfrak{F}}_0 \overset{0''}{\mathfrak{F}}_0\right),\\
&k=2\quad L_2^*({}^1\overset{1}{\mathfrak{F}}_1) = 2\left({}^1\mathfrak{F}_1' + F_1'\,\mathfrak{F}_1'\right) -\\
&\qquad -\left(-2\overset{0'}{\mathfrak{F}}_0 \overset{1'}{\mathfrak{F}}_0 + \overset{0}{\mathfrak{F}}_0 \overset{1''}{\mathfrak{F}}_0 + \overset{0''}{\mathfrak{F}}_0 \overset{1}{\mathfrak{F}}_0\right) - \eta F_1\, {}^1\mathfrak{F}_1'',\quad L_2^*({}^2\mathfrak{F}_1) = 0\\
&k=2\quad L_2^*({}^{11}\mathfrak{F}_{11}) = -\left(-2\overset{0'}{\mathfrak{F}}_0\, {}^1\mathfrak{F}_1' + \overset{0}{\mathfrak{F}}_0\, {}^1\mathfrak{F}_1'' + \overset{0''}{\mathfrak{F}}_0\, {}^1\mathfrak{F}_1\right),\\
&\qquad\quad L_2^*({}^{11}\mathfrak{F}_2) = -\left(\overset{0''}{\mathfrak{F}}_0\, {}^1\mathfrak{F}_1 - \overset{0'}{\mathfrak{F}}_0\, {}^1\mathfrak{F}_1'\right),
\end{aligned}
$$

with boundary conditions

$$
(3.13)\qquad
\begin{aligned}
&\overset{0}{\mathfrak{F}}_0 = \overset{0'}{\mathfrak{F}}_0 = 0,\quad \overset{1}{\mathfrak{F}}_0 = \overset{1'}{\mathfrak{F}}_0 = 0,\ \ldots,\quad \eta = 0,\\
&\overset{0'}{\mathfrak{F}}_0 \to 1,\ \overset{1'}{\mathfrak{F}}_0 \to 0,\ \ldots,\quad \eta \to \infty.
\end{aligned}
$$

In the system (3.12), $L_k$ is an operator of the following type

$$
(3.14)\qquad L_k^* = D_\eta''' + \sigma\eta D_\eta'' - 2\sigma k D_\eta',
$$

where $\sigma = F_0/A^2$. Now we shall determine constants $F_{i_1 \ldots i_k}$ appearing in the above equations. In (2.7) we had $F = \zeta - g_1$ and $\zeta = \zeta(\{g_k\})$. Assuming

$$
(3.15)\qquad \zeta = \zeta_0 + g_1 \zeta_1 + g_1^2 \zeta_{11} + g_2 \zeta_2 + \cdots,
$$

and taking into account that $\zeta = A\left(\dfrac{\partial^2 \mathfrak{F}_0}{\partial \eta^2}\right)_{\eta=0}$ we obtain

$$
(3.16)\qquad \zeta_{i_1 \ldots i_k} = A\left(\overset{i_1 \ldots i_k}{\mathfrak{F}_0}(0)\right)''.
$$

From $F=\zeta-g_1$, (3.15) and (3.11) we have

$$F_0=\zeta_0,\quad F_1=\zeta_1-1,\quad F_{11}=\zeta_{11},\quad F_2=\zeta_2,\ \ldots \tag{3.17}$$

If equations of the system (3.12) are solved, once for all, then for solving a concrete problem remain to seek the functional dependence $\delta_p^*=\delta_p^*(t)$ recpectively $z^*=z^*(t)$. Therefore, we ought to solve the equation (2.7), namely the ordinary differential equation of the first order

$$\frac{dz^*}{dt}=2F(\{g_k\})=2F\left(\frac{\Omega'}{\Omega}z^*,\ \frac{\Omega''}{\Omega}z^{*2},\ \ldots\right). \tag{3.18}$$

With the question of solving the system (3.12) and the equation (3.18) we shall occupy ourselves in the next section. Here, we shall still determine the norming constant $A$. To find it we require that the homogenous parts of equations of the system (3.12) are covered with those of the similar solutions. Thus, we have

$$\sigma=\frac{F_0}{A^2}=2 \tag{3.19}$$

hence taking (3.16) and (3.17) we obtain

$$A=\frac{1}{2}\mathfrak{F}_0^{0''}(0). \tag{3.20}$$

The numerical value of the constant will be determined in the next section when the first equation of (3.12) would be solved.

4. *Simple solutions — one-parameter solutions with assumed linearity.*

As equations of the system (3.12) are of the same form as those in the paper 1, then we can use the same procedure to solve them. Here, we shall only write the explicit forms of solutions which correspond to a one-parameter approximations with assumed linearity. Such solutions we shall call *simple solutions.* Thus, for the case of simple solutions we have that the system (3.12) is reduced to the three equations for determining functions $\mathfrak{F}_0^0(\eta)$, $\mathfrak{F}_0^1(\eta)$ and ${}^1\mathfrak{F}_1(\eta)$. Solving these equations according to the procedure in [1] we obtain

$$\mathfrak{F}_0^0(\eta)=1-g_0(\eta),$$

$$\mathfrak{F}_0^{1'}(\eta)=A^{-2}\left[-g_1(\eta)+\frac{1}{4}g_0(\eta)+\frac{F_1}{16}g_{-1}(\eta)\right], \tag{4.1}$$

$${}^1\mathfrak{F}_1{}'(\eta)=A^{-2}\left[-\frac{1}{2}\left(3+\frac{1}{3}\frac{1}{\Gamma^2}\right)g_1(\eta)+\frac{1}{2}g_0(\eta)\right.$$
$$\left.-\frac{1}{12\,\Gamma}g_{-1/2}(\eta)+\frac{1}{16}g_{-1}(\eta)+\frac{1}{2}g_{1/2}^2(\eta)-\frac{1}{2}g_0(\eta)\,g_1(\eta)\right],$$

where $\Gamma=\Gamma(3/2)=\sqrt{\pi}/2$, and $g_\alpha(\eta)=\dfrac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)}\displaystyle\int_\eta^\infty(\gamma-\eta)^{2\alpha}e^{-\gamma^2}d\gamma.$

Moreover, we can also use the general solution obtained in the paper [2] solving the same type equations to solve the quoted system. It is, however, best to have numerical solutions given in tables, which we are not able to give for the technical reasons.

Now we shall briefly analyse the solving procedure of the equation (3.18). For the one-parameter approximation considering the parameter $g_1$ sufficiently small we have

$$\frac{dz^*}{dt}=2\left(F_0+F_1\frac{\Omega'}{\Omega}z^*\right). \tag{4.2}$$

Hence we have a simple solution

$$z^*=2F_0\Omega^{2F_1}\int\Omega^{-2F_1}dt+C\Omega^{2F_1}. \tag{4.3}$$

From (3.16), (3.17), (3.20) and (4.1) we obtain values of the constants $F_0$ and $F_1$

$$F_0=\frac{1}{2\Gamma^2}=\frac{2}{\pi},\quad F_1=-\frac{2}{3}. \tag{4.4}$$

Hence and from (3.19) we have

$$A^2=\pi. \tag{4.5}$$

If we take into account (4.4) and determine the constant $C$ from the condition of finiteness of $z^*$ at $\Omega=0$, the simple solution (4.3) is reduced to the form convenient for practical considerations

$$z^*=\frac{4}{\pi}\frac{1}{\Omega^{4/3}}\int_0^t\Omega^{4/3}dt, \tag{4.6}$$

respectively

$$g_1=\frac{4}{\pi}\frac{\Omega'}{\Omega^{7/3}}\int_0^t\Omega^{4/3}dt. \tag{4.7}$$

The obtained solution as we have already mentioned is a simple solution which corresponds to the assumed linearity. However, from (3.11) neglecting the influence of the parameters $g_2, g_3, \ldots$ of the set $\{g\}$ we have $F=$ $=\sum_{i=0}^{n}F_{i_0\ldots i_{k-1}}g_1^k$. By the linearity of the given function $F(g_k\}$ we make a mistake. Because of that certain corrections ought to be taken into consideration. Namely, instead of a linear form of the function $F$ one ought to take the following form $F=F_0+g_1F_1+\varepsilon(g_1)$, where $\varepsilon(g_1)$ represents a deviation of the function $F$ from the linearity. If we set $\varepsilon_k=\varepsilon[g_1(t_k)]$ where the time $t_k$ corresponds to the arbitrary division of the considered time interval, and exchange the real distribution $\varepsilon(t)$ by the stepped one, then the equation (4.6) can be written in the form of a recurrent expression

$$\Omega^a(t_k)z^*(t_k)=\Omega^a(t_{k-1})z^*(t_{k-1})+(b+\varepsilon_{k-1})\int_{t_{k-1}}^{t_k}\Omega^a(t)\,dt,\quad a=4/3, \tag{4.8}$$

hence one can easily find $z^*$. Near the values $t_k$ which correspond to small values of the parameter $g_1$ one can use the simple solution (4.6). When the other parameters $g_k$, $k=2, 3, \ldots$ are taken into consideration then new difficulties appear. In that case it is best the equation (3.18) to solve by the computer.

5. *Examples*. Here, we shall solve two elementary examples by means of the use of simple solutions and make comparison with the known results given for instance in [5] and [3]. We consider the following examples:

1. Let a cylinder of the radius $R$ is a started impulsively from the rest with a constant velocity $\Omega_\infty$. Then we have

$$U(x, t)=V(x)\,\Omega(t)=2\,\Omega_\infty \sin\frac{x}{R}. \tag{5.1}$$

From (4.6) we obtain

$$z^* = \frac{4}{\pi}t, \tag{5.2}$$

and from (2.8) we have

$$f_1=V'=\frac{2}{R}\cos\frac{x}{R},\; g_1=0, \text{ and } h_1=\Omega_\infty z^*=\Omega_\infty\frac{4}{\pi}t. \tag{5.3}$$

Now, the velocity profile on account of (3.5), (3.10) and (5.3) has the following form

$$\frac{u}{U}=\mathfrak{F}_0^{0\prime}(\eta)+f_1 h_1\, {}^1\mathfrak{F}_1'(\eta), \tag{5.4}$$

where $\mathfrak{F}_0^{0\prime}(\eta)$ and ${}^1\mathfrak{F}_1'(\eta)$ are given by (4.1). A separation instant we find from the condition $\left(\frac{\partial u}{\partial y}\right)_{y=0}=0$, namely because of (5.4) from

$$\mathfrak{F}_0^{0\prime\prime}(0)+f_1 h_1\, {}^1\mathfrak{F}_1''(0)=0.$$

If we find $\mathfrak{F}_0^{0\prime\prime}(0)$ and ${}^1\mathfrak{F}_1''(0)$ from (4.1) and take into account that the time separation arise as $V'<0$ and begin at the point corresponding max $|V'|$, then we obtain

$$1-2\frac{\Omega_\infty}{R}\left(1+\frac{4}{3\pi}\right)t_s=0,$$

namely

$$t_s=\frac{1}{2\left(1+\frac{4}{3\pi}\right)}\,\frac{R}{\Omega_\infty}=0{,}351\frac{R}{\Omega_\infty}. \tag{5.5}$$

2. Let us now consider the uniformly motion of the same cylinder. Then

$$U(x, t) = V(x)\,\Omega_\infty\, t = 2\,\Omega_\infty \sin\frac{x}{R}\, t.$$

By the same procedure as in the previous example we obtain the following condition for determining separation instants

$$\mathfrak{F}_0^{0''}(0) + g_1 \mathfrak{F}_0^{1''}(0) + g_1 h_1\, {}^1\mathfrak{F}_1''(0) = 0, \tag{5.6}$$

where we have

$$z^* = \frac{12}{7\pi}\, t, \quad f_1 = V' = 2\frac{1}{R}\cos\frac{x}{R}, \quad g_1 = \frac{12}{7\pi}, \quad h_1 = \Omega_\infty \frac{12}{7\pi}\, t^2,$$

respectively

$$\mathfrak{F}_0^{0''}(0) = \frac{2}{\sqrt{\pi}}, \qquad \mathfrak{F}_0^{1''}(0) = \frac{1}{3\sqrt{\pi}}, \qquad {}^1\mathfrak{F}_1''(0) = \frac{1}{2\sqrt{\pi}}\left(1 + \frac{4}{3\pi}\right).$$

From (5.6) for the first separation instant we obtain

$$t_s^2 = \frac{3}{2\left(1 + \dfrac{4}{3\pi}\right)} \frac{R}{\Omega_\infty} = 1{,}07\, \frac{R}{\Omega_\infty}. \tag{5.7}$$

If we compare the results (5.5) and (5.7) with those which are to be found in [5] for respective examples then we have quite a coincidence of them.

6. *Conclusion.* As we have given the method, then we shall in a few words analyse it. The purpose of the method we mentioned in the introduction. Here, we shall remain at the analyse of the solution and the obtained results. The solution (3.5) in the form of series expansion corresponds to that given in the method [1]. In that paper we considered the question of a convergence and concluded that the mentioned series converges rapidly in $x$. As well, we mentioned formally in the paper [3] that it would be already quite sufficient to stop on the third approximation corresponding to the second parameter $f_2$ in the set $\{f_k\}$. The same is shown by Loitsianskii [6]. Next, we have already seen that the parameters $f_k$, $g_k$ and $h_k$ generate the parameters $d_k$. Thus, it means that for a sufficient accuracy we can remain at the second parameter $d_2$. Moreover, our elementary examples show that the one-parameter approximation with the assumed linaerity already gives satisfying results for practical considerations. For that case the closed forms (4.1) and (4.6) respectively (4.7) of solutions are obtained. However, for the more accurate calculations there is a need to take two — or even many-parameter solutions. Because of that it is convenient to have numerical tables of universal functions. We are not able to make them. At the end we still mention that the present method can be extended to the temperature case. It will be the object of the subsequent publication.

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