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THE EXISTENCE AND THE UNICITY OF SOLUTION OF A SYSTEM OF OPERATOR DIFFERENTIAL EOUATIONS

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Introduction

In this paper we shall deal with the system:

$$x_i'(\lambda) = \sum_{j=1}^{m} a_{i,j}(\lambda)\, x_j(\lambda) + b_i(\lambda), \qquad i = 1, 2, \ldots, m \tag{1}$$

where $a_{i,j}(\lambda)$, $b_i(\lambda)$ and $x_i(\lambda)$ are functions which map the interval $[0, \lambda]$ into the field K of Mikusiński operators [4].

We will prove that under some conditions for $a_{i,j}(\lambda)$ and $b_i(\lambda)$ there is one and only one solution of the system (1) satisfying the initial conditions.

In an earlier paper [5] the conditions for the existence and the unicity of the solution of the system (1) are expressed by a relation for maximal element of a class of matrices. The author has also solved [6] a special case of the system (1).

As it is known, the quotient field K of Mikusiński also contains the integral operator l, differential operator s and translation operator. For this reason, the system (1) contains some classes of partial differential equations, integral equations, difference equations and their combinations, all for the numerical functions.

1. Preliminaries

Let C be the commutative algebra of complex-valued functions defined and continuous on the interval $[0, \infty)$. The sums and scalar products are defined in the usual way and the product is defined as the finite convolution $\left(fg = \left\{\int_0^t f(t-u)\, g(u)\, du\right\}\right)$. C is an integral domain under convolution and its field extension is the field K. In K the limit, differentiation and integration are defined [4]. The field K consists of " convolution quotients" $\frac{f}{g}$, where $f \in C$ and $g \in C$, $g \neq 0$.

We shall let $f=\{f(t)\}$ denote the representation of $f(t)$ in C and I is the unit element, $s^0=I$.

Furthermore, we denote by $F_p(t)=t^{-p-1}\Phi(-p, -\sigma; -t^{-\sigma})$, $0<\sigma<1$ $p\geqslant 0$, $F_0=F$, where Φ is the known function of E. M. Wright [11]. This function can be written in the form:

$$F_p(t)=\frac{1}{2\pi i}\int_{Re\, z=0} z^p \exp(tz-z^\sigma)\, dz, \qquad t\geqslant 0$$

and it satisfies the inequality:

$$(2) \qquad |F_p(t)|\leqslant \frac{2}{\sigma}\,\Gamma\left(\frac{p+1}{\sigma}\right)\left(\cos\frac{\pi\sigma}{2}\right)^{-\frac{p+1}{\sigma}}, \qquad 0\leqslant t\leqslant T<\infty.$$

By $C_s(\lambda)$ we denote the set of all functions which map the interval $[0, \lambda]$ into $K: \lambda\to s^\beta w(\lambda)$, $\beta\geqslant 0$, where $w(\lambda)=\{w(\lambda, t)\}$, $w(\lambda, t)$ is a continuous complex-valued function defined on the domain $D: 0\leqslant\lambda\leqslant\Lambda$, $0\leqslant t<\infty$.

$\overline{C}_s(\lambda)$ is the set of functions: $\lambda\to\frac{g(\lambda)}{F}$, where $g(\lambda)=\{g(\lambda, t)\}$, $g(\lambda, t)$ is a continuous complex-valued function defined on D.

We know that $s^\beta=\frac{F_\beta}{F}$, $\beta\geqslant 0$, ; consequently $C_s(\lambda)\subset\overline{C}_s(\lambda)$.

$\overline{C}_s(\lambda)$ will form a commutative algebra where product, sum and scalar product are defined in the usual way, making use of those in K. With the defined sum and scalar product $\overline{C}_s(\lambda)$ is only a vector space.

In $\overline{C}_s(\lambda)$ we have a family of semi-norms N_k:

$$N_k\left(\frac{g(\lambda)}{F}\right)=\underset{(\lambda, t)\in D_k}{\text{Max}}|g(\lambda, t)|$$

where the domain D_k is: $0\leqslant\lambda\leqslant\Lambda$, $0\leqslant t\leqslant k$, $k\in N$.

It is easy to show that $\overline{C}_s(\lambda)$ is a space of B_0 type of Mazur and Orlicz [3]. The topology of $\overline{C}_s(\lambda)$ is finer than the topology by Mikusiński.

When λ is fixed, $\overline{C}_s(\lambda)$ is a subset of K and by definition we have:

$$\nu_k\left(\frac{g(\lambda)}{F}\right)=\underset{0\leq t\leq k}{\text{Max}}|g(\lambda, t)|, \quad k\in N.$$

These two spaces are defined and investigated in [7]. Therefore we will cite here only some results.

Proposition 1. *Let* $(\eta_n(\lambda))\subset\overline{C}_s(\lambda)$ *be a sequence which converges in* $\overline{C}_s(\lambda)$ *to* $\eta(\lambda)\in\overline{C}_s(\lambda)$, *then in* $\overline{C}_s(\lambda)$:

$$\lim_{n\to\infty}\int_0^\Lambda w(u)\,\eta_n(u)\,du=\int_0^\Lambda w(u)\,\eta(u)\,du$$

for every $w(\lambda)=\{w(\lambda, t)\}$, *where* $w(\lambda, t)$ *is a continuous complex-valued function defined on* D. *The integral is in the sense of Mikusiński.*

P r o o f — Let $\Omega_k = \underset{(\lambda, t) \in D_k}{\text{Max}} |w(\lambda, t)|$, $y_n(\lambda) = \eta_n(\lambda) F$ and $y(\lambda) = \eta(\lambda) F$. We have:

$$N_k\left(\int_0^\Lambda w(u)\, \eta_n(u)\, du - \int_0^\Lambda w(u)\, \eta(u)\, du\right)$$

$$\leqslant \underset{0 \leq t \leq k}{\text{Max}} \int_0^\Lambda |w(u, t)|\; |y_n(u, t) - y(u, t)|\, du$$

$$\leqslant \Lambda \Omega_k N_k(\eta_n(\lambda) - \eta(\lambda)).$$

Proposition 2. *Let M be a bounded and equicontinuous subset of the space $\overline{C}_s(\lambda)$. Then lM is compact in $\overline{C}_s(\lambda)$.*

P r o o f. — We shall let lM denote the set of all elements $l\eta(\lambda)$, $\eta(\lambda) \in M$. Since M is bounded, lM is bounded too; then it follows also that the set FlM of numerical functions is uniformly bounded on D_k for every $k \in N$. This set FlM is equicontinuous because:

$$\left|\int_0^t y(\lambda, u)\, du - \int_0^{t_0} y(\lambda_0, u)\, du\right| \leqslant$$

$$\leqslant \int_0^{t_0} |y(\lambda, u) - y(\lambda_0, u)|\, du + \int_{t_0}^t |y(\lambda, u)|\, du$$

$$\leqslant k N_k(\eta(\lambda) - \eta(\lambda_0)) + |t - t_0|\, N_k(\eta(\lambda))$$

and M is equicontinuous.

By the Ascoli's theorem the set FlM of numerical continuous functions defined on D_k, for every $k \in N$, is compact in the set of continuous functions on D_k. Consequently lM is compact in $\overline{C}_s(\lambda)$.

Finally, we shall let $\prod^m \overline{C}_s(\lambda)$ denote the product space of m spaces $\overline{C}_s(\lambda)$ and with Gothic letters, $\mathfrak{y}(\lambda) = (\eta^1(\lambda), \eta^2(\lambda), \ldots, \eta^m(\lambda))$ the points of this product space. $\mathfrak{N}_k(\mathfrak{y}(\lambda)) = \underset{1 \leq i \leq m}{\text{Max}} N_k(\eta^i(\lambda))$ is the semi-norm in $\prod^m \overline{C}_s(\lambda)$.

2. The sums of the finite set of real numbers

The proof of our theorem on system (1) requires the examination of some sums of real numbers.

Consider the set of m^2 positive numbers $\beta_{i,j}$, $1 \leqslant i \leqslant j \leqslant m$. We shall let σ-sum denote a sum of the form:

$$\sum_{k=0}^{\alpha-1} \beta_{i_k, i_{k+1}}, \quad i_\alpha = i_0, \quad i_k \neq i_j \text{ for } k \neq j.$$

The number of σ-sums is finite.

Let δ be the maximum value of the quotients $\frac{\sigma}{\alpha}$. Every sum $\beta_{i_0, i_1} + \beta_{i_1, i_2} + \cdots + \beta_{i_{k-1}, i_k}$, $k > m$ may be expanded into a sum of σ—sums and a remainder P whose number of elements in the sum is less than m, and we have:

$$\sum_{p=0}^{k-1} \beta_{i_p, i_{p+1}} = \sum_{i=1}^{r} \sigma_i + P \leqslant \delta \sum_{i=1}^{r} \alpha_i + P \leqslant \delta k + \gamma$$

where γ is a constant.

In this part we have used some of the results and notations of D. Ž. Đoković [1].

3. The system of differential equations

J. Drobot and Mikusiński [2] proved the unicity of the solution of the system (1) when $a_i(\lambda)$ are independent of λ. J. Mikusiński gave the conditions for the existence of the solution of a differential equation of order n but when the coefficients of this equation are polynomials of s with numerical coefficientes.

Theorem. *Let* $a_{i,j}(\lambda) = s^{\beta_{i,j}} w_{i,j}(\lambda) \in C_s(\lambda)$, $b_i(\lambda) \in \overline{C}_s(\lambda)$ *and* $\delta < 2$, *then the system (1) with the initial condition* $x_i(0) = x_i^0 \in K$, $i = 1, 2, \ldots, m$, *has one and only one solution in* $\overline{C}_s(\lambda)$.

Proof. — Without any restrictions we can suppose that $x_i^0 \in C$ and that $b_i(\lambda) = \{b_i(\lambda, t)\}$ where $b_i(\lambda, t)$ is a continuous function on D.

When we use the properties of integral defined for operator functions, we can show that the system (1) is equivalent to the system of integral equations:

$$x_i(\lambda) = x_i^0 + \sum_{j=1}^{m} \int_0^\lambda a_{i,j}(u)\, x_j(u)\, du + \int_0^\lambda b_i(u)\, du. \tag{3}$$

In order to find the solution of the system (3) we shall conctruct the following sequences:

$$x_i^n(\lambda) = \begin{cases} x_i^0, & 0 \leqslant \lambda \leqslant \Lambda/n \\ x_i^0 + \sum\limits_{i_1=1}^{m} \int\limits_0^{\lambda - \Lambda/n} a_{i, i_1}(\lambda_1)\, x_{i_1}^n(\lambda_1)\, d\lambda_1 + \int\limits_0^\lambda b_i(\lambda_1)\, d\lambda_1, & \Lambda/n < \lambda \leqslant \Lambda), \end{cases} \tag{4}$$

$i = 1, 2, \ldots, m$. We shall introduce some notations to economize in writing. Let $B_i(\lambda) \equiv \int_0^\lambda b_i(u)\, du$ and $\mathbf{A}_i(p, \lambda)$ an integral operator which maps a function $q_{i_1}(\lambda_1) \in \overline{C}_s(\lambda)$ in such a way:

$$\mathbf{A}_i(p, \lambda)\, q_{i_1}(\lambda_1) = \sum_{i_1=1}^{m} \int_{\frac{p\Lambda}{n}}^{\lambda - \Lambda/n} a_{i, i_1}(\lambda_1)\, q_{i_1}(\lambda_1)\, d\lambda_1$$

It is easy to verify that for $\frac{k+1}{n}\Lambda<\lambda\leqslant\frac{k+2}{n}\Lambda,\ j+1\leqslant k\leqslant n-2$:

$$\mathbf{A}_i(j,\lambda)=\sum_{p=j}^{k-1}\mathbf{A}_i\left(p,\frac{p+2}{n}\Lambda\right)+\mathbf{A}_i(k,\lambda). \tag{5}$$

Now the sequences (4) can be written:

$$x_i^n(\lambda)=\begin{cases}x_i^0, & 0\leqslant\lambda\leqslant\Lambda/n\\ x_i^0+\mathbf{A}_i(0,\lambda)\,x_{i_1}^n(\lambda_1)+B_i(\lambda), & \Lambda/n<\lambda\leqslant\Lambda.\end{cases} \tag{6}$$

At first we shall show that the sequences (6), for $\frac{k+1}{n}\Lambda<\lambda\leqslant\frac{k+2}{n}\Lambda$, $1\leqslant k\leqslant n-2$, have also the form:

$$\begin{aligned}x_i^n(\lambda)=x_i^0&+\mathbf{A}_i(0,\lambda)\,x_{i_1}^0+B_i(\lambda)+\mathbf{A}_i(1,\lambda)\,[\mathbf{A}_{i_1}(0,\lambda_1)\,x_{i_2}^0+\\ &+B_{i_1}(\lambda_1)]+\cdots+\mathbf{A}_i(k,\lambda)\,\mathbf{A}_{i_1}(k-1,\lambda_1)\cdots\mathbf{A}_{i_{k-1}}(1,\lambda_{k-1})\times\\ &\times[\mathbf{A}_{i_k}(0,\lambda_k)\,x_{i_{k+1}}^0+B_{i_k}(\lambda_k)].\end{aligned} \tag{7}$$

When $k=0$, we have a trivial case:

$$x_i^n(\lambda)=x_i^0+\mathbf{A}_i(0,\lambda)\,x_{i_1}^0+B_i(\lambda),\qquad i=1,2,\ldots,m.$$

In order to verify (7) for $k=1$, we will use the relation (5).

$$\begin{aligned}x_i^n(\lambda)&=x_i^0+\left[\mathbf{A}_i\left(0,\frac{2}{n}\Lambda\right)+\mathbf{A}_i(1,\lambda)\right]x_{i_1}^n(\lambda_1)+B_i(\lambda)\\ &=x_i^0+\mathbf{A}_i\left(0,\frac{2}{n}\Lambda\right)x_{i_1}^0+\mathbf{A}_i(1,\lambda)\,[x_{i_1}^0+\mathbf{A}_{i_1}(0,\lambda_1)\,x_{i_2}^0+\\ &\quad+B_{i_1}(\lambda_1)]+B_i(\lambda)\\ &=x_i^0+\mathbf{A}_i(0,\lambda)\,x_{i_1}^0+B_i(\lambda)+\mathbf{A}_i(1,\lambda)\,[\mathbf{A}_{i_1}(0,\lambda_1)\,x_{i_2}^0+B_{i_1}(\lambda_1)].\end{aligned}$$

Now we suppose that the relation (7) is true for $\lambda\leqslant\frac{k+1}{n}\Lambda$.

$$\begin{aligned}x_i^n(\lambda)&=x_i^0+\mathbf{A}_i(0,\lambda)\,x_{i_1}^n(\lambda_1)+B_i(\lambda)\\ &=x_i^0+\sum_{p=0}^{k-1}\mathbf{A}_i\left(p,\frac{p+2}{n}\Lambda\right)x_{i_1}^n(\lambda_1)+\mathbf{A}_i(k,\lambda)\,x_{i_1}^n(\lambda_1)+B_i(\lambda)\\ &=x_i^0+\mathbf{A}_i\left(0,\frac{2}{n}\Lambda\right)x_{i_1}^0+\mathbf{A}_i\left(1,\frac{3}{n}\Lambda\right)[x_{i_1}^0+\mathbf{A}_{i_1}(0,\lambda_1)\,x_{i_2}^0+B_{i_1}(\lambda_1)]+\\ &\quad+\cdots+\mathbf{A}_i\left(k-1,\frac{k+1}{n}\Lambda\right)\{x_{i_1}^0+\mathbf{A}_{i_1}(0,\lambda_1)\,x_{i_2}^0+B_{i_1}(\lambda_1)+\mathbf{A}_{i_1}(1,\lambda_1)\times\\ &\quad\times[\mathbf{A}_{i_2}(0,\lambda_2)\,x_{i_3}^0+B_{i_2}(\lambda_2)]+\cdots+\mathbf{A}_{i_1}(k-2,\lambda_1)\cdots\mathbf{A}_{i_{k-3}}(1,\lambda_{k-3})\times\end{aligned}$$

$$\times [\mathbf{A}_{i_{k-2}}(0, \lambda_{k-2}) x^0_{i_{k-1}} + B_{i_{k-2}}(\lambda_{k-2})]] \} + \mathbf{A}_i(k, \lambda) \{ x^0_{i_1} + \mathbf{A}_{i_1}(0, \lambda_1) x^0_{i_2} +$$

$$+ B_{i_1}(\lambda_1) + \mathbf{A}_{i_1}(1, \lambda_1) [\mathbf{A}_{i_2}(0, \lambda_2) x^0_{i_3} + B_{i_2}(\lambda_2)] + \cdots +$$

$$+ \mathbf{A}_{i_1}(k-1, \lambda_1) \ldots \mathbf{A}_{i_{k-2}}(1, \lambda_{k-2}) [\mathbf{A}_{i_{k-1}}(0, \lambda_{k-1}) x^0_{i_k} + B_{i_{k-1}}(\lambda_{k-1})]] \} + B_i(\lambda).$$

We will use once more the relation (5) to sum up the corresponding elements and we come to the relation (7).

The next fact we will prove is: $F_p x_i^n(\lambda) = \{ v_i^n(\lambda, t) \}$ where $v_i^n(\lambda, t)$, $i = 1, 2, \ldots, m$, are bounded sequences on D_q for every $q \in N$. For this reason consider the absolute value of the following function which is continuous on D_q.

$$| F_p \mathbf{A}_i(k, \lambda) \ldots \mathbf{A}_{i_{k-1}}(1, \lambda_{k-1}) [\mathbf{A}_{i_k}(0, \lambda_k) x^0_{i_{k+1}} + B_{i_k}(\lambda_k)] | =$$

$$= \left| F_p \sum_{i_1=1}^{m} \int_{\frac{k\Lambda}{n}}^{\lambda - \Lambda/n} a_{i, i_1}(\lambda_1) \, d\lambda_1 \ldots \sum_{i_k=1}^{m} \int_{\frac{\Lambda}{n}}^{\lambda_{k-1} - \Lambda/n} a_{i_{k-1}, i_k}(\lambda_k) \, d\lambda_k \right. \times$$

$$\left. \times \left[\sum_{i_{k+1}=1}^{m} \int_0^{\lambda_k - \Lambda/n} a_{i_k, i_{k+1}}(\lambda_{k+1}) x^0_{i_{k+1}} \, d\lambda_{k+1} + B_{i_k}(\lambda_k) \right] \right|$$

$$= \left| F_{p+\delta k+\gamma} \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} l^{\delta k + \gamma - \beta_{i, i_1} - \ldots - \beta_{i_{k-1}, i_k}} \int_{\frac{k\Lambda}{n}}^{\lambda - \Lambda/n} d\lambda_1 \ldots \int_{\frac{\Lambda}{n}}^{\lambda_{k-1} - \Lambda/n} w_{i, i_1}(\lambda_1) \ldots \right.$$

$$\left. \ldots w_{i_{k-1}, i_k}(\lambda_k) \, d\lambda_k \left[\sum_{i_{k+1}=1}^{m} \int_0^{\lambda_k - \Lambda/n} a_{i_k, i_{k+1}}(\lambda_{k+1}) x^0_{i_{k+1}} \, d\lambda_{k+1} + B_{i_k}(\lambda_k) \right] \right|$$

$$\leqslant C_q \frac{m^k M_q \Lambda^{(\delta+1)k+\gamma}}{\Gamma^2(k)} \Gamma\left(\frac{\delta k + p + \gamma + 1}{\sigma}\right) \left(\cos \frac{\pi\sigma}{2}\right)^{-\frac{p+1}{\sigma}} = O\left(k^{-\left(2 - \frac{\delta}{\sigma}\right)k}\right)$$

Here we used the relation (2) and we have introduced a constant C_q depending on q and another $M_q = \max_{(\lambda, t) \in D_q} | w_{i,j}(\lambda, t) |$, $1 \leqslant i, j \leqslant m$.

Now it is easy to see that the sequence $(v_i^n(\lambda, t))$ is bounded independently of p fix and positive:

$$| v_i^n(\lambda, t) | = | F_p x_i^n(\lambda) | \leqslant | F_p x_i^0 | + | F_p \mathbf{A}_i(0, \lambda) x^0_{i_1} + F_p B_i(\lambda) | +$$

$$+ | F_p \mathbf{A}_i(1, \lambda) [\mathbf{A}_{i_1}(0, \lambda_1) x^0_{i_2} + B_{i_1}(\lambda_1)] | + \cdots +$$

$$+ | F_p \mathbf{A}_i(k, \lambda) \mathbf{A}_{i_1}(k-1, \lambda_1) \ldots \mathbf{A}_{i_{k-1}}(1, \lambda_{k-1}) [\mathbf{A}_{i_k}(0, \lambda_k) x^0_{i_{k+1}} +$$

$$+ B_{i_k}(\lambda_k)] | \leqslant \sum_{k=0}^{\infty} O\left(k^{-\left(2 - \frac{\delta}{\sigma}\right)k}\right)$$

Our supposition is that $0 < \delta < 2$ and $\sigma \in (0, 1)$ may be chosen in such a manner that the series in last relation converges.

When $p=0$ it follows that $N_q(x_i^n(\lambda))$ is bounded for every q, accordingly m sequences $(x_i^n(\lambda))$ are bounded, related to every seminorm N_q.

We shall give the proof that the elements of the sequence $(x_i^n(\lambda))$ for a fixed i are equicontinuous. Let $\lambda_0 \neq 0$, we can choose n such that $\frac{\Lambda}{n} < \lambda_0$

$$\nu_q(x_i^n(\lambda)-x_i^n(\lambda_0)) \leqslant \nu_q[(\mathbf{A}_i(0,\lambda)-\mathbf{A}_i(0,\lambda_0))\, x^n_{i_i}(\lambda_1)] + \nu_q(B_i(\lambda)-B_i\lambda_0))$$

$$\leqslant \operatorname*{Max}_{0\leq t\leq q} \sum_{i_1=1}^{m} \int_{\lambda_0-\Lambda/n}^{\lambda-\Lambda/n} w_{i,i_1}(\lambda_1)\, F_{\beta_{i,i_1}} x_{i_1}^n(\lambda_1)\, d\lambda_1 \Bigg| + B_i^q\,|\lambda-\lambda_0|,$$

B_i^q is a constant.

We have seen that $(F_p x_i^n(\lambda))$ is a bounded sequence of functions defined on D_q, let this bound be denoted by C, than:

$$\nu_q(x_i^n(\lambda)-x_i^n(\lambda_0)) \leqslant (m M_q C + B_i^q)\,|\lambda-\lambda_0|.$$

When $\lambda=0$, the proof of the last inequality is yet easier.

For m sequences $(x_i^n(\lambda))$ we proved that they are bounded to every seminorm N_q and their elements are equicontinuous. The proposition 2 allows us to choose m subsequences $(l x_i^{n_k}(\lambda))$ which converge on $l x_i(\lambda)$. We will prove that this limit is the solution of the system (3).

As every element of the sequences $(x_i^n(\lambda))$ satisfies the initial conditions, they are also satisfied by the limit $x_i(\lambda)$.

We shall let $\lambda \neq 0$. Starting from the relation (4) we have:

$$l x_i^{n_k}(\lambda) = x_i^0 l + \sum_{i_1=1}^{m} \int_0^{\lambda-\Lambda/n_k} a_{i,i_1}(\lambda_1)\, l x_{i_1}^{n_k}(\lambda_1)\, d\lambda_1 + \int_0^{\lambda} l b_i(\lambda_1)\, d\lambda_1$$

$$(8) \qquad = x_i^0 l + \sum_{i_1=1}^{m} \int_0^{\lambda} a_{i,i_1}(\lambda_1)\, l x_{i_1}^{n_k}(\lambda_1)\, d\lambda_1 + \int_0^{\lambda} l b_i(\lambda_1)\, d\lambda_1 -$$

$$-\sum_{i_1=1}^{m} \int_{\lambda-\Lambda/n_k}^{\lambda} a_{i,i_1}(\lambda)\, l x_{i_1}^{n_k}(\lambda_1)\, d\lambda_1.$$

It remains to show that the last sum in this relation tends to zero when $n_k \to \infty$:

$$N_q\left(\sum_{i_1=1}^{m} \int_{\lambda-\Lambda/n_k}^{\lambda} a_{i,i_1}(\lambda_1)\, l x_{i_1}^{n_k}(\lambda_1)\, d\lambda_1\right) \leqslant C M_q \frac{\Lambda}{n_k}.$$

Now, according to the proposition 1, when we take the limit in the relation (8), we have the solution to system (3) and (1) respectively.

In order to prove the uniqueness of the solution of the system (1), satisfying the initial conditions, we suppose that there exist two solutions satisfying the same initial conditions. The difference of these two solutions is the solution of the system:

$$(9) \qquad x_i(\lambda) = \sum_{i_1=1}^{m} \int_0^{\lambda} a_{i,i_1}(\lambda_1)\, x_{i_1}(\lambda_1)\, d\lambda_1, \qquad i=1, 2, \ldots, m$$

with the initial conditions $x_i(0)=0$, $i=1, 2, \ldots, m$.

After n iteration of the operator from the relation (9) we have:

$$x_i(\lambda) = \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} \int_0^{\lambda} d\lambda_1 \ldots \int_0^{\lambda_{n-1}} a_{i,i_1}(\lambda_1) \ldots a_{i_{n-1},i_n}(\lambda_n) x_{i_n}(\lambda_n) \, d\lambda_n .$$

We know that $x_i(\lambda)$ is bounded for every $i=1, 2, \ldots, m$ as a limit of a bounded sequence, and the inequalities already used imply:

$$N_q(x_i(\lambda)) = O\left(n^{-\left(2-\frac{\delta}{\sigma}\right)n}\right)$$

for every $q \in N$. When $n \to \infty$ we have $x(\lambda) \equiv 0$ the only solution of the system (9) which satisfies the initial condition $x_i(0)=0$, $i=1, 2, \ldots, m$. Hence the theorem is proved.

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