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# ON AN INEQUALITY OF TCHAKALOFF

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**0.** In [4] Tchakaloff showed that Rado's inequality, [1, (12)], can be strengthened if the sequence is assumed to be monotonic. In this note we extend his result to a multiplicative analogue of Rado's inequality, [1, (11)], and to more general means.

**1.** Let $(w)=(w_1, \ldots, w_n)$ denote an $n$-tple of positive numbers and write $W_n=\sum_{k=1}^{n} w_k$. If $(a)=(a_1, \ldots, a_n)$ is another such $n$-tple put

$$A_n=A_n(a;\, w)=\frac{1}{W_n}\sum_{k=1}^{n} a_k w_k,$$

$$G_n=G_n(a;\, w)=\left(\prod_{k=1}^{n} a_k^{\,w_k}\right)^{\frac{1}{W_n}},$$

the weighted arithmetic and geometric means of $(a)$ with weight $(w)$. We will write $A_{n-1}(a;\, w)$ for $A_{n=1}(a';\, w')$ where $(a)'=(a_1, \ldots, a_{n-1})$ etc.

**2.** The following theorem was proved in [3]; we give an alternative proof.

Theorem 1.

(a)

$$Q_n\left\{A_n(a;\, q)-G_n(a;\, p)^{q_nP_n/p_nQ_n}\right\}$$
(2.1)
$$\geqslant Q_{n-1}\left\{A_{n-1}(a;\, q)-G_{n-1}(a;\, p)^{q_nP_{n-1}/p_nQ_{n-1}}\right\}$$

*with equality only when* $G_{n-1}(a;\, p)=a_n^{\,q_nP_{n-1}/p_nQ_{n-1}}$.

(b)

$$P_n\left\{\log\frac{Q_n}{P_n}A_n(a;\, q)-\log\frac{q_n}{p_n}G_n(a;\, p)\right\}$$
(2.2)
$$\geqslant P_{n-1}\left\{\log\frac{Q_{n-1}}{P_{n-1}}A_{n-1}(a;\, q)-\log\frac{q_n}{p_n}G_{n-1}(a;\, p)\right\}$$

*with equality only when* $A_{n-1}(a;\, q)=a_n\dfrac{q_n P_{n-1}}{p_n Q_{n-1}}$.

Proof: Let $\alpha_n(a_1, \ldots, a_n) = A_n(a;q) - G_n(a;p)^{q_n P_n / p_n Q_n}$ and $\alpha_{n-1}(a_1, \ldots, a_{n-1}) = A_{n-1} - G_{n-1}^{q_n P_{n-1}/p_n Q_{n-1}}$. Put $f(x) = \alpha_n(a_1, \ldots, a_{n-1}, x)$, then

$$f(x) = \frac{Q_{n-1}}{Q_n} A_{n-1} + \frac{q_n}{Q_n} x - G_{n-1}^{q_n P_{n-1}/p_n Q_{n-1}} x^{q_n/Q_n},$$

so

$$f'(x) = \frac{q_n}{Q_n}\left(1 - G_{n-1}^{q_n P_{n-1}/p_n Q_{n-1}} x^{-Q_{n-1}/Q_n}\right).$$

Hence we easily see that $f'(x) > 0, = 0, < 0$ according as $x >, =, < G_{n-1}^{q_n P_{n-1}/p_n Q_{n-1}}$. That is to say that at this value of $x$ $f$ has an absolute minimum. So

$$\alpha_n(a_1, \ldots, a_n) > \alpha_n\left(a_1, \ldots, a_{n-1}, G_{n-1}^{q_n P_{n-1}/p_n Q_{n-1}}\right)$$

$$= \frac{Q_{n-1}}{Q_n} \alpha_{n-1}(a_1, \ldots, a_{n-1}),$$

unless $a_n = G_n^{q_n P_{n-1}/p_n Q_{n-1}}$, when we get equality. This proves (2.1).

(b)

A similar proof can be given for (2.2).

**3.** In [4] Tchakaloff showed that particular cases of (2.1) can be improved by assuming $a_n \geqslant \max(a_1, \ldots, a_{n-1})$. We now consider this.

Theorem 2. *If $p_1 q_n \neq p_n q_1$ then inequality (2.1) cannot be improved by assuming $a_n \geqslant \max(a_1, \ldots, a_{n-1})$.*

Proof: Let $a_1 = 1 - \varepsilon$, $a_2 =$ $, \ldots = a_n = 1$. Then with the above notation

$$\text{(3.1)} \qquad \frac{\alpha_n(a_1, \ldots, a_n)}{\alpha_{n-1}(a_1, \ldots, a_{n-1})} = \frac{\left(1 - \dfrac{q_1}{Q_n}\varepsilon\right) - (1-\varepsilon)^{p_1 q_n / p_n Q_n}}{\left(1 - \dfrac{q_1}{Q_{n-1}}\varepsilon\right) - (1-\varepsilon)^{p_1 q_n / p_n Q_{n-1}}}.$$

Letting $\varepsilon \to 0$, the ringht-hand side of (3.1) tends to $\dfrac{Q_{n-1}}{Q_n}$, which completes the proof of Theorem 2.

If follows from this that to improve (2.1) we need to assume $p_1 q_n = p_n q_1$; and if we want a result for all $n$ this is equivalent to assuming $(p)$, $(q)$ are proportional (i.e. for some $\alpha$, $p_k = \alpha q_k$, $1 \leqslant k \leqslant n$), or more simply that $(p) = (q)$.

Theorem 3. *If $n > 2$, $a_n \geqslant \max(a_1, \ldots, a_{n-1})$ then*

$$\text{(3.2)} \qquad \frac{Q_n^2}{Q_n - q_1}\{A_n(a;q) - G(a;q)\} \geqslant \frac{Q_{n-1}^2}{Q_{n-1} - q_1}\{A_{n-1}(a;q) - G_{n-1}(a;q)\},$$

*with equality only when $a_1 = \cdots = a_n$; further this inequality cannot be improved.*

Proof. The last remark follows from (3.1) with $(p)=(q)$; letting $\varepsilon\to 0$ the right-hand side tends to $\dfrac{Q_{n-1}^2(Q_n-q_1)}{Q_n^2(Q_{n-1}-q_1)}$.

The rest of the proof is similar to Tchakaloff's proof in [4] so will not be given in detail. Let

$$\beta(a_1, \ldots, a_n)=Q_n^2(Q_{n-1}-q_1)(A_n-G_n)-Q_{n-1}^2(Q_n-q_1)(A_{n-1}-G_{n-1})$$

and put $g(x)=\beta(a_1 \ldots, a_{n-p}, x, \ldots, x)$. Then

$$g(x)=Q_n^2(Q_{n-1}-q_1)\left(A_{n-p}\frac{Q_{n-p}}{Q_n}+\frac{Q_n-Q_{n-p}}{Q_n}-G_{n-p}{}^{Q_{n-p}/Q_n}x^{\frac{Q_n-Q_{n-p}}{Q_n}}\right)$$

$$-Q_{n-1}^2(Q_{n-1}-q_1)\left(A_{n-p}\frac{Q_{n-p}}{Q_{n-1}}+\frac{Q_{n-1}-Q_{n-p}}{Q_{n-1}}-G_{n-p}{}^{Q_{n-p}/Q_{n-1}}x^{\frac{Q_{n-1}-Q_{n-p}}{Q_{n-1}}}\right)$$

and so

$$g'(x)=Q_n(Q_{n-1}-q_1)(Q_n-Q_{n-p})\left(1-G_{n-p}{}^{Q_{n-p}/Q_n}x^{-Q_{n-p}/Q_n}\right)$$

$$-Q_{n-1}(Q_n-q_1)(Q_{n-1}-Q_{n-p})\left(1-G_{n-p}{}^{Q_{n-p}/Q_{n-1}}x^{-Q_{n-p}/Q_{n-1}}\right).$$

Putting $t^{Q_nQ_{n-1}}=G_{n-p}{}^{Q_{n-p}}x^{-Q_{n-p}}$, $g'(x)=\varphi(t)$, we see that

$$\varphi'(t)=Q_nQ_{n-1}t^{Q_{n-1}-1}\{(Q_n-q_1)(Q_{n-1}-Q_{n-p})t^{q_n}-(Q_{n-1}-q_1)(Q_n-Q_{n-p})\}.$$

It is then easily proved that if $0<t<1$ then $\varphi(t)>0$ and the proof proceeds as in [4].

Since
$$\frac{Q_{n-1}^2(Q_n-q_1)}{Q_n^2(Q_n-q_1)}>\frac{Q_{n-1}}{Q_n},$$

(3.2) is stronger than (2.1). Further if $q_k=1$, $1\leqslant k\leqslant n$, (3.2) reduces to Tchakaloff's inequality.

**4.** We now consider the analogous problem for the inequality (2.2). Since, unlike (2.1), $a_1=\cdots=a_n$ is not a case of equality in (2.2), it not to be expected that results can be obtained by considering approaches to this case, as was done above. Our approach is suggested by the simplest case of equality,

$$a_k=\frac{p_k}{q_k};\quad 1\leqslant k\leqslant n.$$

Theorem 4. *If $(p)$, $(q)$ are not proportional then inequality* (2.2) *cannot be improved by assuming* $\dfrac{a_n q_n}{p_n}\geqslant\max\left(\dfrac{a_1q_1}{p_1},\ldots,\dfrac{a_{n-1}q_{n-1}}{p_{n-1}}\right)$.

Proof. If $\frac{a_1 q_1}{p_1}=1-\varepsilon$, $\frac{a_1 q_2}{p_2}=\cdots=\frac{a_n q_n}{p_n}=1$ then

$$\lim_{\varepsilon\to 0}\frac{\log\frac{Q_n}{P_n}A_n-\log\frac{q_n}{p_n}G_n}{\log\frac{Q_{n-1}}{P_{n-1}}A_{n-1}-\log\frac{q_n}{p_n}G_{n-1}}$$

is just $P_{n-1}/P_n$, which proves the theorem.

So to improve (2.2) we will assume $(p)=(q)$.

Theorem 5. *If $n>2$, $a_n\geqslant\max(a_1,\ldots,a_{n-1})$ then*

$$\frac{Q_n^2}{Q_n-q_1}\{\log A_n(a;\ q)-\log G_n(a;\ q)\}$$
$$\geqslant\frac{Q_{n-1}^2}{Q_{n-1}-q_1}\{\log A_{n-1}(a;\ q)-\log G_{n-1}(a;\ q)\}, \tag{4.1}$$

*with equality only when $a_1=\cdots=a_n$; further this inequality cannot be improved.*

Proof: Putting $(p)=(q)$ in the ratio used in Theorem 4 the limit evaluated there is now equal to $\frac{Q_{n-1}^2(Q_n-q_1)}{Q_n^2(Q_{n-1}-q_1)}$, which demonstrates the last remark.

Let

$$\gamma(a_1,\ldots,a_n)=\frac{(A_n/G_n)^{Q_n^2(Q_{n-1}-q_1)}}{(A_{n-1}/G_{n-1})^{Q_{n-1}^2(Q_n-q_1)}}$$

and put $h(x)=\gamma(a_1,\ldots,a_{n-p},x,\ldots,x)$. Then

$$h'(x)=h(x)\left\{Q_n(Q_{n-1}-q_1)(Q_n-Q_{n-p})\left(\frac{1}{A_n}-\frac{1}{x}\right)\right.$$
$$\left.-Q_{n-1}(Q_n-q_1)(Q_{n-1}-Q_{n-p})\left(\frac{1}{A_{n-1}}-\frac{1}{x}\right)\right\}.$$

If we show $x\geqslant a_{n-p}$ implies $h'(x)\geqslant 0$ then, proceeding as in [4], the proof of (4.1) will follow. Since clearly $h\geqslant 0$ it is sufficient to prove

$$Q_n(Q_{n-1}-q_1)(Q_n-Q_{n-p})\left(\frac{1}{A_n}-\frac{1}{x}\right)$$
$$\geqslant Q_{n-1}(Q_n-q_1)(Q_{n-1}-Q_{n-p})\left(\frac{1}{A_{n-1}}-\frac{1}{x}\right).$$

Since $x\geqslant a_{n-p}$ implies both $x\geqslant A_n$ and $x\geqslant A_{n-1}$, this last inequality is equivalent to

$$Q_n(Q_{n-1}-q_1)(Q_n-Q_{n-p})(x-A_n)$$
$$\geqslant Q_{n-1}(Q_n-q_1)(Q_{n-1}-Q_{n-p})(x-A_{n-1}),$$

which in turn is equivalent to

$$(Q_{n-1}-q_1)(Q_n-Q_{n-p})\geqslant(Q_n-q_1)(Q_{n-1}-Q_{n-p}).$$

Simple calculations verify this last inequality, and complete the proof.

By the remark following Theorem 3, (4.1) is stronger that (2.2). Further if $p_k = 1$, $1 \leqslant k \leqslant n$ then (4.1) becomes

$$\left(\frac{A_n}{G_n}\right)^{\frac{n^2}{n-1}} \geqslant \left(\frac{A_{n-1}}{G_{n-1}}\right)^{\frac{(n-1)^2}{n-2}}, \tag{4.2}$$

a multiplicative analogue of Tchakaloff's inequality and an improvement on [1, (11)].

**5.** In [1] the multiplicative analogue of Rado's inequality was extended to more general means. The ideas of Tchakaloff can also be extended to these means; as the proofs are the same but much more tedious we will only state the results.

If $1 \leqslant r \leqslant n$ put

$$E(r) = E(r, n) = E(r, n; a) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant n} \prod_{j=1}^{n} a_{i_j},$$

$$P(r) = P(r, n) = P(r, n; a) = \binom{n}{r}^{-1} E(r).$$

$E(r)$ is the rth elementary symmetric function of $(a)$, and $P(r)$ the rth symmetric mean of $(a)$. As in 0 we will write $E(r, n-1) = E(r, n-1, a)$ for $E(r, n-1, a')$ etc.

The following results are known, [1, 2];

(i) if $s < t$ then $P^t(s) \geqslant P^s(t)$ with equality only when $a_1 = \cdots = a_n$;

(ii) if $(a) = (a_1, \ldots, a_{n-q}, x, \ldots, x)$ then

$$P(s, n) = \sum_{t=a}^{r} \lambda(s, t) P(s-t, n-q) x^t,$$

where $u = \max(s-n+q, 0)$, $v = \min(s, q)$, and

$$\lambda(s, t) = \binom{n-q}{s-t}\binom{q}{t} \Big/ \binom{n}{s};$$

(iii) if $1 \leqslant r < k < n$ then

$$\frac{P^{k+1}(r, n)}{P^r(k+1, n)} \geqslant \frac{P^k(r, n-1)}{P^r(k, n-1)}, \tag{5.1}$$

with equality only when $a_1 = \cdots = a_n$. It is (5.1) that we will strengthen; if $r = 1$, $k = n-1$, then (5.1) reduces to the multiplicative analogue of Radó's Theorem, [1].

Theorem 6. *If $n > 2$, $a_n \geqslant \max(a_1, \ldots, a_{n-1})$ then*

$$(k+1)\log P(r, n) - r\log P(k+1, n) \tag{5.2}$$

$$\geqslant \frac{(k+1)(n-1)^2}{(k-1)n^2}\{k\log P(r, n-1) - r\log P(k, n-1)\},$$

*with equality only when $a_1 = \cdots = a_n$; further this inequality cannot be improved.*

The last remark is proved by putting $a_1 = 1-\varepsilon$, $a_2 = \cdots = a_n = 1$ and evaluating $\lim\limits_{\varepsilon \to 0} \dfrac{(k+1)\log P(r,\ n) - r \log P(k+1,\ n)}{k \log P(r,\ n-1) - r \log P(k,\ n-1)}$; its value is just $\dfrac{(k+1)(n-1)}{(k-1)n^2}$. The rest is proved as in Theorem 5, making use in particular of (ii) above.

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