PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 9 (23), 1969. pp. 5—7

ON MULTI-VALUED FUNCTIONS

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(Communicated June 23, 1967)

The aim of this paper is to show that all multi-valued linear functions on vector space to vector space can be reduced to single-valued linear functions.

A *multi-valued* function is a correspondence which assigns to each point of a set X a subset of a set Y.

We denote multi-valued functions by capital letters Γ, F, etc., and single-valued functions by small letters γ, f, etc.

In [1] is assumed that for some $x \in X$ it can be $\Gamma(x) = \varnothing$.

A function Γ on a space X to a space Y is called *continuous* at x' if and only if for all open set W containing $\Gamma(x')$ there exists an open set V containing x' such that $\Gamma(x) \subset W$ for all $x \in V$. Γ is continuous on X if it is continuous for all $x \in X$.

First of all, we shall examine the structure of multi-valued linear functions defined in [1].

Let X and Y be two vector spaces. A multi-valued function $\Gamma : X \to Y$ is said to be *linear* provided that:

1° for $\left.\begin{matrix} y \in \Gamma(x) \\ y' \in \Gamma(x') \end{matrix}\right\} \Rightarrow y + y' \in \Gamma(x + x')$;

2° if $y \in \Gamma(x)$ and α scalar $\Rightarrow \alpha y \in \Gamma(\alpha x)$.

For linear functions is also acceptable $\Gamma(x) = \varnothing$ for some $x \in X$.

A multi-valued linear function Γ we call *trivial* if $\Gamma(x) = \varnothing$ for all $x \in X$. Γ is *constant function* if $\Gamma(x) = \Gamma(0)$ for all $x \in X$.

Theorem 1. *If multi-valued linear function Γ is not trivial, then $\Gamma(0) \neq \varnothing$.*

Proof. For some $x \in X$, it is, then, valid $\Gamma(x) \neq \varnothing$. But then, for $y \in \Gamma(x)$ we obtain $0 = 0 \cdot y \in \Gamma(0 \cdot y) = \Gamma(0)$, and $\Gamma(0)$ is non-void.

This justifies the above definition of constant function.

Theorem 2. *If Γ is linear and $\Gamma(x) \neq \varnothing$, then $\Gamma(-x) \neq \varnothing$.*

Proof. Since $\Gamma(x) \neq \varnothing$ there exists at least one element y in $\Gamma(x)$.

But then, because of linearity $-y = (-1)\, y \in \Gamma((-1)\, x) = \Gamma(-x)$, and $\Gamma(-x)$ is non-void.

Theorem 3. *Let $\Gamma : X \to Y$ be linear multi-valued function on a vector space X to a vector space Y; then $\Gamma(0)$ is a vector subspace of Y.*

Proof. For $y, y' \in \Gamma(0)$ we have, according to the definition of linearity, $y + y' \in \Gamma(0+0) = \Gamma(0)$, and the set $\Gamma(0)$ is closed with respect to the operation $+$. Associativity is included since $\Gamma(0)$ is a part of the Abelian group Y. For $\lambda \in R$ (= set of scalars, real line for example) and $y \in \Gamma(0)$ we have $\lambda y \in \Gamma(\lambda 0) = \Gamma(0)$. So $\lambda = 0$ and $y \in \Gamma(0)$ implies $0 = 0 \cdot y \in \Gamma(0)$. In the same way: $\lambda = -1$ and $y \in \Gamma(0)$ implies $(-1 \cdot y = -y \in \Gamma((-1)\,0) = \Gamma(0)$, i.e. $\Gamma(0)$ contains $-y$ with y.

The following theorem gives the structure of multi-valued linear functions.

Theorem 4. *If Γ is multi-valued linear function on a vector space X to a vector space Y, then for all $x \in X$ (for which $\Gamma(x) \neq \varnothing$) the set $\Gamma(x)$ is an equivalence class with respect to the equivalence relation ρ defined in the following way: $a \rho b$ if and only if $a - b \in \Gamma(0)$.*

Proof. Suppose $\Gamma(x) \neq \varnothing$. Take $y, y' \in \Gamma(x)$. Since Γ is linear and taking into account the theorem 2. we have $-y' \in \Gamma(-x)$. But then $y - y' \in \Gamma(x + (-x)) = \Gamma(0)$ what was to be proved.

According to the last theorem, there is no difference, in the algebraic meaning of the word, between multi-valued linear functions on X to Y and single-valued functions on X to Y/ρ, where ρ is the equivalence relation defined in theorem 4. But if one takes into account topologies of X and Y, and considers continuous linear multi-valued functions these two functions are different. In other words, if $\Gamma: X \to Y$ is linear multi-valued function and if the single-valued function $\gamma: X \to Y/\Gamma(0)$ is defined so that $\gamma(x) = \Gamma(x) \in Y/\Gamma(0)$, then γ can be continuous (supposing that the topology of $Y/\Gamma(0)$ is quotient topology) while Γ is discontinuous. We shall show it by the following

Example. Take $X = Y = R$ (= real numbers with usual topology). Consider multi-valued function $\Gamma: R \to R$ defined in the following way: $\Gamma(x) = ([x], [x]+1)$ for $x \neq [x]$ and $\Gamma(x) = \{x\}$ for $x = [x]$, ($[x]$ is the greatest integer contained in x). This function makes a partition of R the elements of which are: open intervals $(k-1, k)$, (denote them by D_k), and $\{k\}$, where k is an integer. We shall show that Γ is not continuous. Let $x = k$ and $0 < \varepsilon < 1$. Take the open set $G = k - \varepsilon, k + \varepsilon)$. G contains $\Gamma(k) = \{k\}$. But no one of the neighbourhoods of k can be transferred into G, since $\Gamma(G(k)) = (k-1, k+1)$ for all $G(k)$ contained in $(k-1, k+1)$.

The quotient topology of the space $\mathcal{D} = \{D_k, \{k\}\}$ has D_k as isolated points and the basic neighbourhood of $\{k\}$ is $\{D_k, \{k\}, D_{k+1}\}$. The induced function $\gamma: R \to \mathcal{D}$, defined so that $\gamma(x) = \Gamma(x)$, is evidently continuous. So, the continuity of γ in the quotient topology does not imply the continuity of Γ as multi-valued function.

To remove this discrepancy we shall equip $Y/\Gamma(0)$ with another topology. That is *choice topology* introduced in [2] in the following way. Let X be topological space and $\mathcal{D}$ a partition of X, i.e. a family of disjoint subsets of X which covers X (every element of X belongs to one and only one element of the partition). Let $\varphi: \mathcal{D} \to X$ be function on $\mathcal{D}$ to X defined so that $\varphi(D) \in D$ for all $D \in \mathcal{D}$. Such a function is known by the name *choice function*. Let Z be the family of all choice functions φ. By the *choice topology* we mean the coarsest topology on $\mathcal{D}$ for which all the choice functions φ are continuous.

A multi-valued function $\Gamma: X \to Y$ is called semi-single-valued if and only if $\Gamma(x_1) \cup \Gamma(x_2) \neq \varnothing$ implies $\Gamma(x_1) = \Gamma(x_2)$.

Theorem 6. *Let X and Y be topological spaces, ρ equivalence relation on Y and let $\mathcal{D}=Y/\rho$ be equipped with choice topology. Let Γ be semi-single--valued function on X to Y and γ single-valued function on X to $\mathcal{D}$, defined in such way that $\gamma(x)=\Gamma(x)$. Then the continuity of Γ implies the continuity of γ and conversely.*

Proof. Consider $\gamma(x)$ as an element of $\mathcal{D}$ which is equipped with choice topology, and let $O_{\mathcal{D}}$ be an open set containing $\gamma(x)$. The interior of the subset of Y whose elements belong to D (denote it by 0), for $D\in O_{\mathcal{D}}$ contains $\Gamma(x)$, or, otherwise $O_{\mathcal{D}}$ would not contain $\gamma(x)$ and be open. But since Γ is continuous, for O containing the set $\Gamma(x)$, there exists an open neighbourhood V of x such that for all $x'\in V$ the set $\Gamma(x')\subset 0$. Consider now that subset of $\mathcal{D}$, the elements of which are those equivalence classes D which satisfy condition $D\cup O\neq\varnothing$. Denote that subset of $\mathcal{D}$ by $O'_{\mathcal{D}}$. According to the definition of choice topology, $O'_{\mathcal{D}}$ is open in $\mathcal{D}$. We have $O'_{\mathcal{D}}\subseteq O_{\mathcal{D}}$. But then, $x'\in V$ implies $\gamma(x')\in 0'$, that is $\gamma(x')\in 0$ and γ is continuous function on X to $\mathcal{D}$ equipped with choice topology.

Conversely, let $\gamma: X\to\mathcal{D}$ be continuous. Let U be an open set in Y containing $\Gamma(x)$, $x\in X$ Consider the subset $U_{\mathcal{D}}$ of $\mathcal{D}$ which elements D satisfy condition $D\subset U$. Using choice function φ such that for those D which satisfy condition $0\cap D\neq\varnothing$ and $\mathbf{C}O\cap D\neq\varnothing$ it is valid $\varphi(D)\in\mathbf{C}O\cap D$, we have that $\varphi^{-1}(0)$ is open in $\mathcal{D}$. Put $\varphi^{-1}(0)=O_{\mathcal{D}}$. Evidently $\varphi(x)\in O_{\mathcal{D}}$. Since γ is continuous there exists an open set V in X containing x such that $\gamma(x')\in O_{\mathcal{D}}$ for all $x'\in V$. But then $\Gamma(x')\subset 0$ for all $x'\in V$, so that Γ is also continuous.

Using previous facts concerning linear multi-valued functions and theorem 6., we can express the following main result of the paper.

Theorem 7. *Let $\Gamma: X\to Y$ be linear multi-valued function on a topological vector space X to a topological vector space Y and let $\mathcal{D}=Y/\Gamma(0)$ be equipped with choice topology and let $\gamma: X\to\mathcal{D}$ be defined so that $\gamma(x)=\Gamma(x)$. Then: the continuity of Γ implies continuity of γ and conversely.*

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