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ASYMPTOTIC BEHAVIOUR OF THE MAXIMAL ELEMENT OF A MATRIX

D. Ž. Đoković

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1. Let $\hat{R}$ be the additive group of real numbers with an additional element $-\infty$ such that $(-\infty)+a=a+(-\infty)=-\infty$, $-\infty<a$ for any real number a. Let $A=(a_{ij})$, $a_{ij}\in\hat{R}$, $B=(b_{jk})$, $b_{jk}\in\hat{R}$ be $m\times n$ and $n\times p$ matrices respectively. We define $A * B=U=(u_{ik})$ as $m\times p$ matrix such that

$$u_{ik}=\max_{j}(a_{ij}+b_{jk}).$$

The operation $*$ is associative. If $C=(c_{kr})$, $c_{kr}\in\hat{R}$ is $p\times q$ matrix then (i, r)-th entry of $A * B * C$ is

$$\max_{j,k}(a_{ij}+b_{jk}+c_{kr}).$$

In the sequel $A=(a_{ij})$, $a_{ij}\in\hat{R}$ is a fixed $n\times n$ matrix. We set

$$A * A=2 * A,\quad A * A * A=3 * A,\quad \ldots$$

$$\mu(A)=\max_{i,j} a_{ij}.$$

In [1] the problem of asymptotic behaviour of $\mu(k * A)$ as $k\to+\infty$ was raised. We shall prove that $\mu(k * A)=ak+b_k$ where a is fixed and b_k is bounded.

2. *A path* P is a sequence $P=(i_0, i_1, \ldots, i_\alpha)$ of integers taken among $1, 2, \ldots, n$. For this path P we define

$$l(P)=\alpha,\quad \sigma(P)=a_{i_0, i_1}+a_{i_1, i_2}+\cdots+a_{i_{\alpha-1}, i_\alpha}.$$

A path $P=(i_0, i_1, \ldots, i_\alpha)$ is *simple* if $i_k\neq i_j$ for $k\neq j$. *A simple cycle* is a path $Z=(i_0, i_1, \ldots, i_\alpha)$ such that $i_0=i_\alpha$, $i_k\neq i_j$ for $k\neq j$ otherwise. For any simple cycle Z we define

$$\varepsilon(Z)=\sigma(Z)/l(Z).$$

The number of all simple cycles Z is finite. Therefore there is a simple cycle Z_0 such that

$$\varepsilon_0=\varepsilon(Z_0)=\max_{Z}\varepsilon(Z).$$

We shall write $l_0=l(Z_0)$.

If $\varepsilon_0 = -\infty$ then $\varepsilon(Z) = -\infty$ for all Z. If $l(P) \geqslant n$ then P contains some Z as a subsequence, which implies that $\sigma(P) = -\infty$. Hence, $\mu(k * A) = -\infty$ for $k \geqslant n$.

Now we assume that $\varepsilon_0 > -\infty$. Any path P can be decomposed into simple cycles and a simple path. For instance, let $n=5$ and $P=(3, 2, 1, 4, 5, 4, 5, 2, 4)$. This path contains the simple cycle $Z_1 = (5, 4, 5)$. After deleting $(5, 4)$ from P we get the path $(3, 2, 1, 4, 5, 2, 4)$. This path contains the simple cycle $(2, 1, 4, 5, 2) = Z_2$. After deleting $(2, 1, 4, 5)$ we get the simple path $P_1 = (3, 2, 4)$. In this example we write $P = Z_1 + Z_2 + P_1$. We notice that this decomposition is not unique.

In the general case we have

$$P = k_1 Z_1 + \cdots + k_r Z_r + P_1,$$

where $k_1, \ldots, k_r$ are positive integers and P_1 is a simple path. We must have $l(P_1) < n$. For $l(P) = k$ we get

$$k = k_1 l(Z_1) + \cdots + k_r l(Z_r) + l(P_1).$$

We obtain

$$\begin{aligned}\sigma(P) &= k_1 \sigma(Z_1) + \cdots + k_r \sigma(Z_r) + \sigma(P_1)\\ &= k_1 l(Z_1)\, \varepsilon(Z_1) + \cdots + k_r l(Z_r)\, \varepsilon(Z_r) + \sigma(P_1)\\ &\leqslant \varepsilon_0 [k_1 l(Z_1) + \cdots + k_r l(Z_r)] + \sigma(P_1)\\ &= \varepsilon_0 (k - l(P_1)) + \sigma(P_1)\\ &\leqslant k\, \varepsilon_0 + M_1\end{aligned}$$

where M_1 is a constant independent of k. Since $\mu(k * A) = \max \sigma(P)$ where max is taken over all P such that $l(P) = k$ we get

$$\mu(k * A) \leqslant k\, \varepsilon_0 + M_1.$$

If $k = ml_0 + p$ where m and p $(< l_0)$ are non-negative integers we take $P = mZ_0 + P_2$ where P_2 is a part of Z_0 and $l(P_2) = p$. Then

$$\begin{aligned}\mu(k * A) &\geqslant \sigma(P)\\ &= m\, \sigma(Z_0) + \sigma(P_2)\\ &= ml_0\, \varepsilon_0 + \sigma(P_2)\\ &= k\, \varepsilon_0 - p\, \varepsilon_0 + \sigma(P_2)\\ &\geqslant k\, \varepsilon_0 - M_2\end{aligned}$$

where M_2 is a constant independent of k.

We have proved the following

Theorem. *If* $\varepsilon_0 = -\infty$ *then* $\mu(k * A) = -\infty$ *for* $k \geqslant n$. *If* $\varepsilon_0 > -\infty$ *then*

$$k\, \varepsilon_0 - M_2 \leqslant \mu(k * A) \leqslant k\, \varepsilon_0 + M_1 \quad (k = 1, 2, 3, \ldots)$$

where M_1 *and* M_2 *are constants independent of* k.

3. *An example.* Assume that A has the following form

$$A=\begin{bmatrix} -\infty & 0 & -\infty & -\infty & \cdots & -\infty \\ -\infty & -\infty & 0 & -\infty & & -\infty \\ -\infty & -\infty & -\infty & 0 & & -\infty \\ \vdots & & & & & \\ -\infty & -\infty & -\infty & -\infty & & 0 \\ a_1 & a_2 & a_3 & a_4 & & a_n \end{bmatrix}.$$

If we set

$$Z_1=(1, 2, 3, \ldots, n-1, n, 1),$$
$$Z_2=(2, 3, \ldots, n-1, n, 2),$$
$$\vdots$$
$$Z_n=(n, n),$$

then

$$\varepsilon(Z_r)=\frac{a_r}{n+1-r} \quad (r=1, 2, \ldots, n).$$

For all remaining simple cycles Z we have $\varepsilon(Z)=-\infty$. Hence,

$$\varepsilon_0=\max_r \frac{a_r}{n+1-r}.$$

REFERENCE

[1] B. Stanković, *L'élément maximal d'une matrice*, Publications de l'Institut mathématique (Beograd), **6** (**20**) (1966), 23—24.