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SOME INEQUALITIES WITH CONVEX FUNCTIONS

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Starting with an elementary inequality in 1., an integral inequality is proved in 2. and can be considered as a special case of inequalities treated in [3]. This inequality is used as lemma in proving Karamata's inequality in 3. ([2] and [1]) as well as a somewhat sharpened form of Steffensen's inequality ([3] and [1]). Note that our proofs are not only much shorter but show that these two inequalities are related being both the easy consequences of the inequality from 2.

1. Let $\{a_\nu\}$ and $\{b_\nu\}$ be two finite sequences of non-negative numbers such that

$$a_1+a_2+\cdots+a_\nu \geqslant b_1+b_2+\cdots+b_\nu, \qquad \nu=1,\ 2,\ \ldots,\ n-1$$
$$a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n$$

what is usually written as $\{a_\nu\}\succ\{b_\nu\}$.

Let $\{a_\nu\}\succ\{b_\nu\}$ *and let* $\{\alpha_\nu\}$ *be a* $\left\{\begin{array}{l}\textit{increasing}\\ \textit{decreasing}\end{array}\right.$ *sequence of numbers, than*

$$\sum_{\nu=1}^{n}\alpha_\nu a_\nu \left\{\begin{array}{l}\leqslant\\ \geqslant\end{array}\right. \sum_{\nu=1}^{n}\alpha_\nu b_\nu .$$

Proof. Let $\{\alpha_\nu\}$ be increasing, then from

$$(a_1-b_1)+\cdots+(a_n-b_n)=0,$$

it follows that

$$\alpha_n(a_n-b_n)+\alpha_n\sigma_{n-1}=0,$$

where $\sigma_{n-1}=(a_1-b_1)+\cdots+(a_{n-1}-b_{n-1})$. Since $\sigma_{n-1}\geqslant 0$, we have

$$\alpha_n(a_n-b_n)+\alpha_{n-1}(a_{n-1}-b_{n-1})+\alpha_{n-1}\sigma_{n-2}\leqslant 0,$$

where $\sigma_{n-2}=(a_1-b_1)+\cdots+(a_{n-2}-b_{n-2})$. Continuing in this way, we get the above inequality. When $\{\alpha_\nu\}$ is decreasing the proof goes similarly.

2. For two non-negative functions g_1 and g_2 defined on $[0,\ a]$, we write $g_1\succ g_2$ to denote that

$$\int_0^x g_1\,dt\geqslant\int_0^x g_2\,dt \quad\text{and}\quad \int_0^a g_1\,dt=\int_0^a g_2\,dt, \quad (x\in[0,\ a])$$

Now, we can prove the following:

If f is integrable and $\begin{cases} \textit{increasing} \\ \textit{decreasing} \end{cases}$ *on the interval* $[0, a]$, *then*

$$\int_0^a fg_1\,dx \begin{cases} \leqslant \\ \geqslant \end{cases} \int_0^a fg_2\,dx.$$

Proof. Let us put $G_\tau(x) = \int_0^x g_\tau(t)\,dt$, $x \in [0, a]$ and $\tau = 1, 2$. If f is a step function which takes the value α_τ on the interval $[x_\tau, x_{\tau+1}]$, then

$$\int_0^a fg_1\,dx = \sum \alpha_\tau [G_1(x_{\tau+1}) - G_1(x_\tau)]$$

and in virtue of 1., when f is increasing

$$\int_0^\alpha fg_1\,dx \leqslant \sum \alpha_\tau [G_2(x_{\tau+1}) - G_2(x_\tau)] = \int_0^a fg_2\,dx.$$

So, the above inequality is proved when f is a step function. Since the set of step functions is dense in $L_{[0,a]}$, the inequality holds for arbitrary f.

3. Karamata's inequality: Let $\{a_\nu\}$ and $\{b_\nu\}$ be monotonously decreasing and such that $\{a_\nu\} \succ \{b_\nu\}$. Let f is a monotonously increasing function on $[0, a_1]$. Let us put

$$A(x) = \sum_{\nu=1}^n m\{[0, x] \cap [0, a_\nu]\}, \quad B(x) = \sum_{\nu=1}^n m\{[0, x] \cap [0, b_\nu]\}$$

(mS is the measure of the set S). Then

$$A(x) \leqslant B(x), \qquad A(a) = B(a)$$

and $A'(x)$ and $B'(x)$ exist everywhere except in a finite set of points. So by the inequality in 2,

$$(*) \qquad \int_0^{a_1} f dA(x) \geqslant \int_0^{a_1} f dB(x).$$

Putting $F(x) = \int_0^x f dx$, where $F(x)$ can be an arbitrary continuous convex function when f is arbitrary, we get

$$\int_0^{a_1} f dA(x) = n \int_0^{a_n} f dx + (n-1) \int_{a_n}^{a_{n-1}} f dx + \cdots + 1, \quad \int_{a_2}^{a_1} f dx$$

$$= F(a_1) + F(a_2) + \cdots + F(a_n).$$

Now the inequality (*) can be written as

$$F(a_1) + F(a_2) + \cdots + F(a_n) \geqslant F(b_1) + F(b_2) + \cdots + F(b_n),$$

what is known as Karamata's inequality.

4. *Let* $0 \leqslant g(x) \leqslant 1$ *and* $f \searrow$ *on* $[0, a]$. *Then*

$$\int_0^a fg\,dx \leqslant F\left(\int_0^a g\,dx\right)$$

where $F(x) = \int_0^x f(t)\,dt$.

Proof. Let $c = \int_0^a g\,dx$, then $0 < c \leqslant a$ and let

$$\tilde{g}(x) = \begin{cases} 1, & x \in [0, c] \\ 0, & x \in [c, a] \end{cases}.$$

Since $\tilde{g} \succ g$, we have by 2.,

$$\int_0^c f\,dx = \int_0^a f\tilde{g}\,dx \geqslant \int_0^a fg\,dx$$

what is Steffensen's inequality.

Now we state and prove somewhat generalized form of Steffensen's inequality:

Let $g(x) \geqslant 0$ *and* $f \searrow$ *on* $[0, a]$. *Then*

$$\int_0^a fg\,dx \leqslant \gamma \int_0^{c/\gamma} f\,dx,$$

where $\gamma = \sup\{g(x);\ x \in [0, a]\}$.

Proof. Let

$$\tilde{g} = \begin{cases} \gamma, & x \in \left[0, \dfrac{c}{\gamma}\right] \\ 0, & x \in \left[\dfrac{c}{\gamma}, a\right] \end{cases},$$

then $\tilde{g} \succ g$.

REFERENCES

[1] Beckenbach E. and Bellman R., *Inequalities*, Springer-Verlag, 1961.

[2] Karamata J., *Sur une inégalité relative aux fonctions convexes*, Publ. Math. Univ. Belgrade **1** (1932).

[3] Fan K., Lorentz G., *An integral inequality*, Am. Math. Monthly, 61 (1954).

[4] Steffensen J. E., *Bounds of certain trigonometric integrals*, Tenth Scand. Math. Congress 1946, 181—186. Copenhagen, Gjellerups, 1947.