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AN APPROXIMATE SOLUTION OF THE BOUNDARY LAYER ON A BODY STARTED FROM CERTAIN PRECEDING NON-STEADY MOTIONS

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1. Introduction. — For the motion of a body started from certain preceding non-steady motions, we derived [1] equations of the additional boundary layer as follows:

$$(1.1)\quad \begin{cases} \dfrac{\partial u_d}{\partial t_1}+u_d\dfrac{\partial u_d}{\partial x}+v_d\dfrac{\partial u_d}{\partial y}+u_s\dfrac{\partial u_d}{\partial x}+u_d\dfrac{\partial u_s}{\partial x}+v_s\dfrac{\partial u_d}{\partial y}+v_d\dfrac{\partial u_s}{\partial y}= \\ \qquad =\dfrac{\partial U_d}{\partial t_1}+U_d\dfrac{\partial U_d}{\partial x}+U_s\dfrac{\partial U_d}{\partial x}+U_d\dfrac{\partial U_s}{\partial x}+\nu\dfrac{\partial^2 u_d}{\partial y^2} \\ \qquad\qquad \dfrac{\partial u_d}{\partial x}+\dfrac{\partial v_d}{\partial y}=0 \end{cases}$$

with boundary and initial conditions:

$$u_d=v_d=0,\quad \text{for}\quad y=0;\qquad u_d=U_d(x,\, t_1),\quad \text{for}\quad y=\infty;$$
$$u_d=v_d=0\quad \text{for}\quad t_1=0.$$

where

t — time that elapsed since the beginning of preceding motion,

T — instant at which additional motion takes place,

t_1 — duration of additional motion,

(u_s, v_s) — projections of velocity in the „preceding" boundary layer which continues formally its development for $t\geqslant T$,

(u_d, v_d) — projections of velocity in the additional boundary layer,

x — longitudinal coordinate, as measured from the front stagnation point along the body's contour,

y — transverse coordinate,

ρ — density of the fluid,

ν — kinematic viscosity.

In accordance with procedure developed in [1], solutions of the boundary layer are given by the following expressions:

$$u = u_s + u_d$$

$$v = v_s + v_d$$

In order to solve equations (1.1), a process of successive approximations was formulated. Thus for the first approximations [1], the following equations were obtained:

$$(1.2)\quad \begin{cases} \dfrac{\partial u_0}{\partial t_1} - \nu \dfrac{\partial^2 u_0}{\partial y^2} = \dfrac{\partial U_d}{\partial t_1} + \dfrac{\partial}{\partial x}[U_d (U_s - u_s)] \\ \dfrac{\partial u_0}{\partial x} + \dfrac{\partial v_0}{\partial y} = 0 \\ u_0 = 0, \quad y = 0; \qquad u_0 = U_d(x, t_1), \quad y = \infty . \end{cases}$$

$$(1.3)\quad \begin{cases} \dfrac{\partial u_1}{\partial t_1} - \nu \dfrac{\partial^2 u_1}{\partial y^2} = U_d \dfrac{\partial U_d}{\partial x} + \dfrac{\partial}{\partial x}(u_s U_d) - v_s \dfrac{\partial u_0}{\partial y} - \\ -(u_s + u_0) \dfrac{\partial u_0}{\partial x} - v_0 \dfrac{\partial u_0}{\partial y} - u_0 \dfrac{\partial u_s}{\partial x} - v_0 \dfrac{\partial u_s}{\partial y} \\ \dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = 0 \\ u_1 = 0, \quad y = 0; \qquad u_1 = 0, \quad y = \infty . \end{cases}$$

While solving these equations considerable difficulties arise from the presence of functions u_s and v_s in the right-hand sides of these equations. These functions depend upon the transverse „y" — direction only implicitly through the non-steady variable $\eta = \dfrac{y}{2\sqrt{\nu t}}$. It was possible to show [1] that the value $\eta = 2$ replaces the theoretical value on the upper boundary of the boundary layer $\eta = \infty$, with a minimum discrepancy of only 0.44%. Then a solution was found (1) which corresponds to the interval $0 \leqslant \eta \leqslant 2$. A transformation had to be carried out of the function „u_s" into new variables of the additional boundary layer, „t_1" and $\bar{\eta} = \dfrac{y}{2\sqrt{\nu t_1}}$ by expansions in series and retention of a number of terms. This limitation of the interval η was reflected in difficulties which arose when working out the velocity profile in order to obtain that the velocity in the boundary layer at its upper boundary has precisely the value which the external potential velocity had at the same position. The „completion" of calculations of the boundary layer, in the sense given above, will be achieved by investigating also the interval $2 \leqslant \eta \leqslant \infty$. As we have noted, here $u_s \approx U_s$, where U_s is the external preceding potential velocity. But where $u_s \approx U_s$, we have $v_s \approx 0$, the more so since transverse motions are always negligible.

Because of the subsequent conditions it should be noted that the case $u_s \approx U_s$ may occur even very closely to the contour of the body, depending upon the duration of the motion itself.

By using these conclusions from (1.2) and (1.3) it is possible to simplify the equation for the additional boundary layer:

$$(1.4)\quad \begin{cases} \dfrac{\partial u_0}{\partial t_1} - \nu \dfrac{\partial^2 u_0}{\partial y^2} = \dfrac{\partial U_d}{\partial t_1} \\ \dfrac{\partial u_0}{\partial x} + \dfrac{\partial v_0}{\partial y} = 0 \\ u_0 = 0, \quad y = 0; \qquad u_0 = U_d(x, t_1), \quad y = \infty . \end{cases}$$

$$(1.5)\quad \begin{cases} \dfrac{\partial u_1}{\partial t_1} - \nu \dfrac{\partial^2 u_1}{\partial y^2} = U_d \dfrac{\partial U_d}{\partial x} + \dfrac{\partial}{\partial x}(U_s U_\alpha) - \\ \qquad - (U_s + u_0) \dfrac{\partial u_0}{\partial x} - v_0 \dfrac{\partial u_0}{\partial y} - u_0 \dfrac{\partial U_s}{\partial x} \\ \qquad \dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = 0 \\ u_1 = 0, \quad y = 0; \qquad u_1 = 0, \quad y = \infty . \end{cases}$$

2. Additional impulse following preceding motion started impulsively. — A cylindrical body was started impulsively from rest $[U_s = U(x)]$. At an instant, it was given an additional impulse. The equation (1.4) in the first approximation now becomes

$$\frac{\partial u_0}{\partial t_1} - \nu \frac{\partial^2 u_0}{\partial y^2} = 0$$

with the following boundary conditions:

$$u_0 = 0, \quad y = 0; \qquad u_0 = U(x), \quad y = \infty .$$

If the solution of the preceding equation is sought in the form of

$$(2.1)\qquad u_0 = U(x)\, \zeta_0'(\bar{\eta})$$

we obtain for the unknown function $\zeta_0'(\bar{\eta})$ the equation

$$\zeta_0''' + 2\bar{\eta}\, \zeta_0'' = 0$$

with the following boundary conditions:

$$\zeta_0(0) = \zeta_0'(0) = 0, \qquad \zeta_0'(\infty) = 1 .$$

Its solution fulfilling the given boundary conditions is

$$(2.2)\qquad \zeta_0'(\bar{\eta}) = Erf\, \bar{\eta} = \frac{2}{\sqrt{\pi}} \int_0^{\bar{\eta}} e^{-\gamma^2}\, d\gamma .$$

From the second of equations (1.4) it is possible to obtain „v_0":

$$(2.3)\qquad v_0 = -2\sqrt{\nu t_1}\, U' \zeta_0(\bar{\eta})$$

Thus, the first of the equations (1.5) yields

$$\frac{\partial u_1}{\partial t_1}-\nu\frac{\partial^2 u_1}{\partial y^2}=UU'(3-2\zeta_0'-\zeta_0'^2+\zeta_0\zeta_0''). \tag{2.4}$$

Now, if the solution of this equation is sought in the form of

$$u_1=t_1 UU'\zeta_1'(\bar\eta) \tag{2.5}$$

we obtain from (2.4) as follows

$$\zeta_1'''+2\bar\eta\zeta_1''-4\zeta_1'=-4(3-2\zeta_0'-\zeta_0'^2+\zeta_0\zeta_0'')$$

and

$$\zeta_1'''+2\bar\eta\zeta_1''-4\zeta_1'=-4(P_1\, Erf^2\bar\eta+P_2\, Erf\,\bar\eta+P_3) \tag{2.6}$$

where

$$P_1=-1,\qquad P_2=\frac{2}{\sqrt{\pi}}\bar\eta e^{-\bar\eta^2}-2,$$

$$P_3=\frac{2}{\pi}e^{-2\bar\eta^2}-\frac{2}{\pi}e^{-\bar\eta^2}+3.$$

Seek the particular integral of this non-homogeneous differential equation in the form

$$\zeta_{1p}'(\bar\eta)=X(\bar\eta)\, Erf^2\bar\eta+Y(\bar\eta)\, Erf\,\bar\eta+S(\bar\eta) \tag{2.7}$$

The substitution of (2.7) into (2.6) for the unknown coefficients — functions $X(\bar\eta)$, $Y(\bar\eta)$, $S(\bar\eta)$ leads to the differential equations:

$$\left\{\begin{aligned} & X''+2\bar\eta X'-4X=4 \\ & Y''+2\bar\eta Y'-4Y=-\frac{8}{\sqrt{\pi}}X'e^{-\bar\eta^2}-4P_2 \\ & S''+2\bar\eta S'-4S=-\frac{8}{\pi}Xe^{-2\bar\eta^2}-\frac{4}{\sqrt{\pi}}Y'e^{-\bar\eta^2}-4P_3 \end{aligned}\right. \tag{2.8}$$

The solution of the first equation belonging to the recursive system (2.8) is:

$$X(\bar\eta)=K_1(1+2\bar\eta^2)-1$$

where K_1 is an arbitrary constant.

When this value is introduced into the right-hand side of second of the equations (2.8), the following is obtained:

$$Y''+2\bar\eta Y'-4Y=\left(-\frac{32}{\sqrt{\pi}}K_1-\frac{8}{\sqrt{\pi}}\right)\bar\eta e^{-\bar\eta^2}+8.$$

One of the solutions of this equation, which corresponds to the present problem, runs as follows:

$$Y(\bar\eta)=\left(\frac{4}{\sqrt{\pi}}K_1+\frac{1}{\sqrt{\pi}}\right)\bar\eta e^{-\bar\eta^2}-2$$

Finally, by solving the third of the equations (2.8)

$$S'' + 2\bar{\eta} S' - 4S = \left[\left(\frac{16}{\pi} K_1 + \frac{8}{\pi}\right)\bar{\eta}^2 - \left(\frac{24}{\pi} K_1 + \frac{4}{\pi}\right)\right] e^{-2\bar{\eta}^2} + \frac{8}{\pi} e^{-\bar{\eta}^2} - 12$$

we show that the solution of this equation in a closed form can be obtained only if the value of the constant K_1 is

$$K_1 = \frac{1}{2}.$$

Thus, we come eventually to the corresponding solutions of (2.8):

$$\text{(2.9)} \quad \begin{cases} X(\bar{\eta}) = \bar{\eta}^2 - \dfrac{1}{2} \\ Y(\bar{\eta}) = \dfrac{3}{\sqrt{\pi}} \bar{\eta} e^{-\bar{\eta}^2} - 2 \\ S(\bar{\eta}) = \dfrac{2}{\pi} e^{-2\bar{\eta}^2} - \dfrac{4}{3\pi} e^{-\bar{\eta}^2} + 3 . \end{cases}$$

Hence, the particular integral (2.7) becomes completely definite. Since the particular solutions of the homogeneous part of the differential equation (2.6) are

$$\zeta'_{1h}(\bar{\eta}) = 1 + 2\bar{\eta}^2$$

$$\zeta'_{1h}(\bar{\eta}) = \frac{1}{4}(1 + 2\bar{\eta}^2)\, Erf\, \bar{\eta} + \frac{1}{2\sqrt{\pi}} \bar{\eta} e^{-\bar{\eta}^2}$$

it is possible to formulate the general solution of the initial equation

$$\text{(2.10)} \quad \zeta'_1(\bar{\eta}) = C_1(1 + 2\bar{\eta}^2) + C_2\left[\frac{1}{4}(1 + 2\bar{\eta}^2)\, Erf\, \bar{\eta} + \frac{1}{2\sqrt{\pi}} \bar{\eta} e^{-\bar{\eta}^2}\right] + \zeta'_{1p}(\bar{\eta}) .$$

With boundary conditions

$$\zeta'_1(0) = 0, \qquad \zeta'_1(\infty) = 0$$

we obtain the following values for the constants:

$$C_1 = -\frac{2}{3\pi} - 3, \qquad C_2 = \frac{8}{3\pi} + 10.$$

The addition of (2.1) and (2.5) results in the velocity of the additional boundary layer:

$$\text{(2.11)} \quad u_d = U\zeta'_0(\bar{\eta}) + t_1 UU' \zeta'_1(\bar{\eta}) .$$

Since the velocity of the preceding boundary layer is known [3]:

$$u_s = U \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\gamma^2} d\gamma = Uf'_1(\eta)$$

the total velocity in the boundary layer during additional motion is

$$u = Uf_1'(\eta) + U\zeta_0'(\bar{\eta}) + t_1\, UU'\zeta_1'(\bar{\eta}). \tag{2.12}$$

The universal functions (2.2) and (2.10) have been worked out [1] and their graphs are shown in Fig. 1.

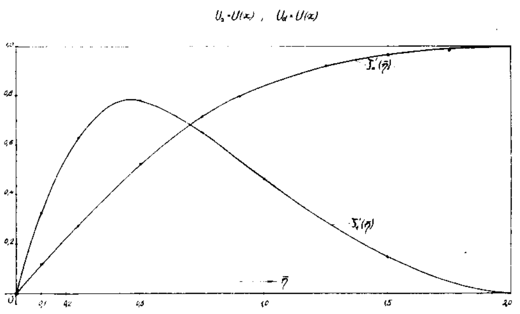

Fig. 1

It is interesting to apply (2.12) in the same instants and on the same places along the contour of the circular cylinder as in the case of the exact solution [1] which corresponds to the interval $0 \leqslant \eta \leqslant 2$. Thus, it would be possible to check whether the values obtained agree or disagree with those obtained by another method, as well as to follow the asymptotic development of the velocity profile, as the distance from the body contour increases. Consider a circular cylinder, having radius $R = 50$ cm, given an additional impulse ($U_\infty = 10$ cm/s) after the preceding impulse ($U_\infty = 10$ cm/s). We calculate now the velocity in the boundary layer, by applying the formula (2.12) to several points on the contour at different instants. Here we give only graphs.

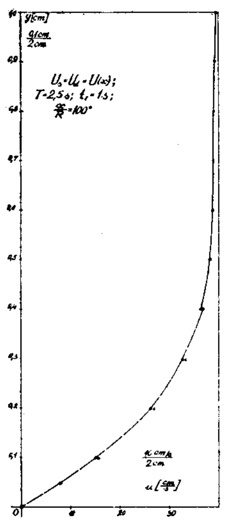

Fig. 2

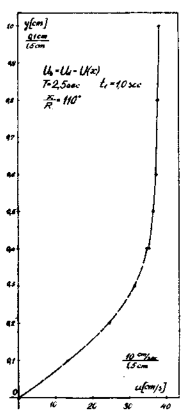

Fig. 3

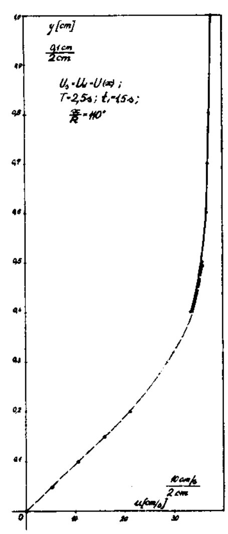

Fig. 4

In Figs. 2, 3 and 4, the lower parts of the velocity profiles were worked out by the „principal solution" corresponding to the interval $0 < \eta \leqslant 2$ [1]. The agreement of our solution with this „principal" solution is highly remarkable. The maximum discrepancy of the velocity values at the point $y = 0.4$ cm in all the three cases in these graphs does not exceed 0.25%. Moreover, our solution (2.12) continues to give satisfactory results in the entire area up to the body contour itself, as suggested by Fig. 4. This is rather significant since the solutions (2.2) and (2.10) are far simpler with respect to the corresponding solutions for the interval $0<\eta\leqslant 2$.

3. Uniformly accelerated motion following a preceding impulse. — A cylindrical body started to motion by an impulse was given at an instant $t_1=0$ an additional uniformly accelerated motion.

The equation for the first approximation is

$$\frac{\partial u_0}{\partial t_1}-\nu\frac{\partial^2 u_0}{\partial y^2}=W$$

$$u_0=0,\quad y=0;\qquad u_0=t_1 W,\quad y=\infty.$$

Its solutions should be sought in the form of

$$u_0=t_1 W\zeta_0'(\bar{\eta}). \tag{3.1}$$

When (3.1) is substituted in the above equation, a differential equation

$$\zeta_0'''+2\bar{\eta}\zeta_0''-4\zeta_0'=-4$$

will be obtained for the determination of the function, the boundary conditions being

$$\zeta_0(0)=\zeta_0'(0)=0,\qquad \zeta_0'(\infty)=1.$$

Its solution, which fulfil these conditions, is

$$\zeta_0'(\bar{\eta})=(1+2\bar{\eta}^2)\,Erf\bar{\eta}+\frac{2}{\sqrt{\pi}}\bar{\eta}\,e^{-\bar{\eta}^2}-2\bar{\eta}^2. \tag{3.2}$$

From the equation of continuity for the first approximation we find the component „v_0":

$$v_0=-2t_1\sqrt{\nu t_1}\,W'\zeta_0(\bar{\eta})$$

and, thus, the equation for the second approximation becomes

$$\frac{\partial u_1}{x\,t_1}-\nu\frac{\partial^2 u_1}{\partial y^2}=t_1^2 WW'(1+\zeta_0\zeta_0''-\zeta_0'^2)+t_1(UW'+U'W)(1-\zeta_0'). \tag{3.3}$$

If the solution of this equation is sought in the form of

$$u_1=t_1^3 WW'\zeta_1'(\bar{\eta})+t_1^2(UW'+U'W)\zeta_2'(\bar{\eta}) \tag{3.4}$$

we obtain the following differential equations:

$$\left\{\begin{array}{l}\zeta_1'''+2\bar{\eta}\zeta_1''-12\zeta_1'=-4(1+\zeta_0\zeta_0''-\zeta_0'^2)\\ \zeta_1(0)=\zeta_1'(0)=0,\quad \zeta_1'(\infty)=0\end{array}\right. \tag{3.5}$$

$$\left\{\begin{array}{l}\zeta_2'''+2\bar{\eta}\zeta_2''-8\zeta_2'=-4(1-\zeta_0')\\ \zeta_2(0)=\zeta_2'(0)=0,\quad \zeta_2'(\infty)=0.\end{array}\right. \tag{3.6}$$

Solutions in a closed form have been found for these equations [1] but we shall not discuss them here. These solutions are:

$$
\begin{aligned}
&\zeta_1'(\bar{\eta}) = C_1\left((1+6\bar{\eta}^2+4\bar{\eta}^4+\frac{8}{15}\bar{\eta}^6\right)+C_2\left[\frac{1}{384}\left(1+6\bar{\eta}^2+4\bar{\eta}^4+\right.\right.\\
&\left.+\frac{8}{15}\bar{\eta}^6\right)(1-\operatorname{Erf}\bar{\eta})-\frac{1}{720\sqrt{\pi}}\left(\bar{\eta}^5+7\bar{\eta}^3+\frac{33}{4}\bar{\eta}\right)e^{-\bar{\eta}^2}\Big]+\Big[K_1\Big(1+\\
&+6\bar{\eta}^2+4\bar{\eta}^4+\frac{8}{15}\bar{\eta}^6\Big)-\frac{4}{3}\bar{\eta}^2-2\bar{\eta}^2-\frac{2}{3}\Big]\operatorname{Erf}^2\bar{\eta}+\Big\{\Big[\Big(\frac{44}{5\sqrt{\pi}}K_1-\\
(3.7)\quad &-\frac{7}{3\sqrt{\pi}}\Big)\bar{\eta}+\Big(\frac{112}{15\sqrt{\pi}}K_1-\frac{8}{3\sqrt{\pi}}\Big)\bar{\eta}^3+\frac{16}{15\sqrt{\pi}}K_1\bar{\eta}^5\Big]e^{-\bar{\eta}^2}+\frac{8}{3}\bar{\eta}^4+4\bar{\eta}^2-\\
&-\frac{16}{15\sqrt{\pi}}\bar{\eta}+\frac{2}{3}\Big\}\operatorname{Erf}\bar{\eta}+\frac{1}{9\pi}(2\bar{\eta}^4+\bar{\eta}^2+8)e^{-2\bar{\eta}^2}+\Big(\frac{8}{3\sqrt{\pi}}\bar{\eta}^3+\frac{7}{3\sqrt{\pi}}\bar{\eta}-\\
&-\frac{16}{15\sqrt{\pi}}\Big)e^{-\bar{\eta}^2}-\frac{4}{3}\bar{\eta}^4-2\bar{\eta}^2+\frac{16}{15\sqrt{\pi}}\bar{\eta}
\end{aligned}
$$

$$C_1=-\frac{5}{12\pi},\qquad C_2=\frac{1024}{15\pi}+288;\qquad K_1=\frac{5}{12}.$$

$$
\begin{aligned}
&\zeta_2'(\bar{\eta}) = C_1\left(1+4\bar{\eta}^2+\frac{4}{3}\bar{\eta}^4\right)+C_2\left[\frac{1}{32}\left(1+4\bar{\eta}^2+\frac{4}{3}\bar{\eta}^4\right)\left(1-Erf\bar{\eta}\right)-\right.\\
(3.8)\quad &\left.-\frac{1}{24\sqrt{\pi}}\left(\frac{5}{2}\bar{\eta}+\bar{\eta}^3\right)e^{-\bar{\eta}^2}\right]-(1+2\bar{\eta}^2)\,Erf\bar{\eta}-\frac{2}{\sqrt{\pi}}\bar{\eta}\,e^{-\bar{\eta}^2}+(1+2\bar{\eta}^2)
\end{aligned}
$$

$$C_1=0,\qquad C_2=-32.$$

The sum of the expressions (3.1) and (3.4) gives the velocity of the additional boundary layer:

$$u_d=t_1W\zeta_0'(\bar{\eta})+t_1^3\,WW'\zeta_1'(\bar{\eta})+t_1^2(UW'+U'W)\,\zeta_2'(\bar{\eta}) \tag{3.9}$$

The total velocity in the boundary layer is

$$u=Uf_1'(\bar{\eta})+t_1W\zeta_0'(\bar{\eta})+t_1^2(UW'+U'W)\,\zeta_2'(\eta)+t_1^3\,WW'\zeta_1'(\bar{\eta}). \tag{3.10}$$

The universal functions (3.2), (3.7) and (3.8) have been plotted in Fig. 5. As an example, we shall calculate now the boundary layer on a circular cylinder with the radius R, which was started impulsively $\left(U=2\,U_\infty\sin\frac{x}{R}\right)$ and given, at a later instant, an additional constantly accelerated motion $\left(W=2\,V_0\sin\frac{x}{R}\right)$. The numerical data are: $R=50$ cm.

$$U_\infty=10\ \text{cm/s},\qquad V_0=10\ \text{cm/s}^2.$$

Fig. 5 shows that the agreement of this solution with the preceding one is quite satisfactory. The discrepancy of the velocity values at $y = 0{,}4$ cm is only 0.17%.

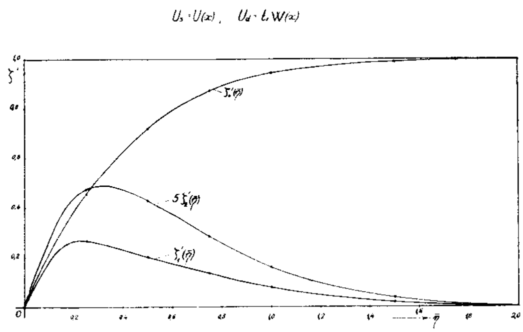

Fig. 5

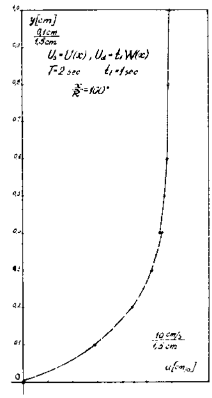

Fig. 6

4. An impulse following a constantly accelerated motion. — A cylindrical body moves with constant acceleration $(U_s = t\,W)$ from rest. At an instant, the body is given an additional impulse $(U_d = U)$. The equation (1.4) for the first approximation, now, becomes:

$$\frac{\partial u_0}{\partial t_1} - \nu \frac{\partial^2 u_0}{\partial y^2} = 0$$

the boundary conditions being:

$$u_0 = 0, \quad y = 0; \qquad u_0 = U(x), \quad y = \infty .$$

If the solution of this equation is sought in the form

$$u_0 = U(x)\,\zeta_0'(\bar{\eta}) \tag{4.1}$$

the function $\zeta_0'(\bar{\eta})$ will have the value (2.2).

The equation for the second approximation (1.5) now becomes

$$\frac{\partial u_1}{\partial t_1} - \nu \frac{\partial^2 u_1}{\partial y^2} = UU' P_1(\bar{\eta}) + T(UW' + U'W) P_2(\bar{\eta}) + \\ + t_1 (U'W + UW') P_3(\bar{\eta}) \tag{4.2}$$

where the known functions are:

$$P_1(\bar{\eta}) = 1 - \zeta_0'^2 + \zeta_0 \zeta_0'', \qquad P_2 = P_3 = 1 - \zeta_0' . \tag{4.3}$$

If the solution of (4.2) is sought in the form

$$u_1 = t_1 [UU' \zeta_1'(\bar{\eta}) + T(UW' + U'W) \zeta_2'(\bar{\eta})] + t_1^2 (UW' + U'W) \zeta_3'(\bar{\eta}) \tag{4.4}$$

the following differential equations will be obtained:

$$\begin{cases} \zeta_1''' + 2\bar{\eta}\,\zeta_1'' - 4\zeta_1' = -4P_1(\bar{\eta}) \\ \zeta_2''' + 2\bar{\eta}\,\zeta_2'' - 4\zeta_2' = -4P_2(\bar{\eta}) \\ \zeta_3''' + 2\bar{\eta}\,\zeta_3'' - 8\zeta_3' = -4P_3(\bar{\eta}) \end{cases} \tag{4.5}$$

with the following boundary conditions:

$$(4.6)\qquad \begin{cases} \zeta_1(0)=\zeta_1'(0)=\zeta_1'(\infty)=0 \\ \zeta_2(0)=\zeta_2'(0)=\zeta_2'(\infty)=0 \\ \zeta_3(0)=\zeta_3'(0)=\zeta_3'(\infty)=0. \end{cases}$$

The solutions of these equations are:

$$\zeta_1'=C_1(1+2\bar{\eta}^2)+C_2\left[\frac{1}{4}(1+2\bar{\eta}^2)\, Erf\,\bar{\eta}+\frac{1}{2\sqrt{\pi}}\,\bar{\eta}\, e^{-\bar{\eta}^2}\right]+$$

$$+\left(\bar{\eta}^2-\frac{1}{2}\right) Erf^2\,\bar{\eta}+\frac{3}{\sqrt{\pi}}\,\bar{\eta}\, e^{-\bar{\eta}^2}\, Erf\,\bar{\eta}+\frac{2}{\pi}e^{-2\bar{\eta}^2}-\frac{4}{3\pi}\, e^{-\bar{\eta}^2}+1$$

$$C_1=-\frac{2}{3\pi}-1,\qquad C_2=\frac{8}{3\pi}+2;$$

$$\zeta_2'=C_1(1+2\bar{\eta}^2)+C_2\left[\frac{1}{4}(1+2\bar{\eta}^2)\, Erf\,\bar{\eta}+\frac{1}{2\sqrt{\pi}}\,\bar{\eta}\, e^{-\bar{\eta}^2}\right]-Erf\,\bar{\eta}+1$$

$$C_1=-1,\qquad C_2=4;$$

$$\zeta_3'=C_1\left(1+4\bar{\eta}^2+\frac{4}{3}\bar{\eta}^4\right)+C_2\left[\frac{1}{32}\left(1+4\bar{\eta}^2+\frac{4}{3}\bar{\eta}^4\right)(1-Erf\,\bar{\eta})-\right.$$

$$\left.-\frac{1}{24\sqrt{\pi}}\left(\frac{5}{2}\,\bar{\eta}+\bar{\eta}^3\right)e^{-\bar{\eta}^2}\right]-\frac{1}{2}\, Erf\,\bar{\eta}+\frac{1}{2}.$$

$$C_1=0,\qquad C_2=-16.$$

The sum of functions (4.1) and (4.4) determines the velocity of the additional boundary layer. Since, according to Blasius [2], the velocity of the preceding boundary layer is given by the following expression:

$$u_s=(T+t_1)\, W(x)\, f_1'(\eta)$$

the resulting velocity in the boundary layer will be

$$(4.7)\qquad \begin{aligned} u=(T+t_1)\, Wf_1'(\eta)+U\zeta_0'(\bar{\eta})+t_1\,[UU'\zeta_1'(\bar{\eta})+ \\ +T(UW'+U'W)\zeta_2'(\bar{\eta})]+t_1^2(UW'+U'W)\,\zeta_3'(\bar{\eta}). \end{aligned}$$

5. Uniformly accelerated motion following a preceding motion started with a constant acceleration. — Here, the functions U_s and U_α are

$$U_s=tW(x),\qquad U_d=t_1\, W(x),$$

and the equation (1.4) becomes

$$\frac{\partial u_0}{\partial t_1}-\nu\frac{\partial^2 u_0}{\partial y^2}=W.$$

Its solution, in view of the initial condition, can be found in the form of

$$(5.1)\qquad u_0=t_1\, W(x)\,\zeta_0'(\bar{\eta})$$

where $\zeta_0'(\bar{\eta})$ has the value (3.2). Hence, the equation for the second approximation (1.5) is

$$\frac{\partial u_1}{\partial t_1} - \nu \frac{\partial^2 u_1}{\partial y^2} = t_1 2TWW' P_1(\bar{\eta}) + t_1^2 WW' P_2(\bar{\eta})$$

where

$$P_1(\bar{\eta}) = 1 - \zeta_0', \qquad P_2(\bar{\eta}) = 3 - 2\zeta_0' - \zeta_0'^2 + \zeta_0 \zeta_0''.$$

If the solution for this equation is sought in the form of

$$u_1 = t_1^2 2TWW'\zeta_1'(\bar{\eta}) + t_1^3 WW'\zeta_2'(\bar{\eta}) \tag{5.2}$$

we obtain the equation

$$\left\{ \begin{array}{l} \zeta_1''' + 2\bar{\eta}\zeta_1'' - 8\zeta_1' = -4P_1(\bar{\eta}) \\ \zeta_2''' + 2\bar{\eta}\zeta_2'' - 12\zeta_2' = -4P_2(\bar{\eta}) \end{array} \right. \tag{5.3}$$

with boundary conditions:

$$\left\{ \begin{array}{l} \zeta_1(0) = \zeta_1'(0) = \zeta_1'(\infty) = 0 \\ \zeta_2(0) = \zeta_2'(0) = \zeta_2'(\infty) = 0. \end{array} \right. \tag{5.4}$$

The solutions of the equation (5.3) are given in a closed form

$$\zeta_1'(\bar{\eta}) = C_1\left(1 + 4\bar{\eta}^2 + \frac{4}{3}\bar{\eta}^4\right) + C_2\left[\frac{1}{32}\left(1 + 4\bar{\eta}^2 + \frac{4}{3}\bar{\eta}^4\right)(1 - Erf\,\bar{\eta}) - \right.$$
$$\left. - \frac{1}{24\sqrt{\pi}}\left(\frac{5}{2}\bar{\eta} + \bar{\eta}^3\right)e^{-\bar{\eta}^2}\right] - (1 + 2\bar{\eta}^2)\,Erf\,\bar{\eta} - \frac{2}{\sqrt{\pi}}\bar{\eta}\,e^{-\bar{\eta}^2} + 2\bar{\eta}^2 + 1$$

$$C_1 = 0, \qquad C_2 = -32.$$

$$\zeta_2'(\bar{\eta}) = C_1\left(1 + 6\bar{\eta}^2 + 4\bar{\eta}^4 + \frac{8}{15}\bar{\eta}^6\right) + C_2\left[\frac{1}{384}\left(1 + 6\bar{\eta}^2 + 4\bar{\eta}^4 + \right.\right.$$
$$\left.\left. + \frac{8}{15}\bar{\eta}^6\right)(1 - Erf\,\bar{\eta}) - \frac{1}{720\sqrt{\pi}}\left(\bar{\eta}^5 + 7\bar{\eta}^3 + \frac{33}{4}\bar{\eta}\right)e^{-\bar{\eta}^2}\right] +$$
$$+ \left[K_1\left(1 + 6\bar{\eta}^2 + 4\bar{\eta}^4 + \frac{8}{15}\bar{\eta}^6\right) - \frac{4}{3}\bar{\eta}^4 - 2\bar{\eta}^2 - \frac{2}{3}\right] Erf\,\bar{\eta}^2 +$$
$$+ \left\{\left[\left(\frac{44}{5\sqrt{\pi}}K_1 - \frac{7}{3\sqrt{\pi}}\right)\bar{\eta} + \left(\frac{112}{15\sqrt{\pi}}K_1 - \frac{8}{3\sqrt{\pi}}\right)\bar{\eta}^3 + \frac{16}{15\sqrt{\pi}}K_1\bar{\eta}^5\right]e^{-\bar{\eta}^2} + \right.$$
$$\left. + \frac{B}{3}\bar{\eta}^4 + 2\bar{\eta}^2 - \frac{16}{15\sqrt{\pi}}\bar{\eta} - \frac{1}{3}\right\} Erf\,\bar{\eta} + \frac{1}{9\pi}(2\bar{\eta}^4 + \bar{\eta}^2 + 8)e^{-2\bar{\eta}^2} +$$
$$+ \left(\frac{8}{3\sqrt{\pi}}\bar{\eta}^3 + \frac{1}{3\sqrt{\pi}}\bar{\eta} - \frac{16}{15\pi}\right)e^{-\bar{\eta}^2} - \frac{4}{3}\bar{\eta}^4 + \frac{16}{15\sqrt{\pi}}\bar{\eta} + 1$$

$$C_1 = -K_1 = -\frac{5}{12}, \qquad C_2 = \frac{1024}{15\pi} - 224.$$

Therefore, with the universal functions obtained as described above, the velocity in the boundary layer becomes completely defined

$$(5.5) \qquad u=(T+t_1)\,W f_1'(\bar{\eta})+t_1\,W\zeta_0'(\bar{\eta})+t_1^2\,2\,TWW'\,\zeta_1'(\bar{\eta})+ + t_1^3\,2\,WW'\,\frac{1}{2}\,\zeta_2'(\bar{\eta}).$$

6. Power series accelerated motion following an impulse. — If a cylindrical body was moving after having been started by an impulse $U_s=U(x)$ and thereupon was given an additional motion specified by $U_d=At^{\alpha}W(x)$, from (1.4) it follows that

$$(6.1) \qquad \frac{\partial u_0}{\partial t_1}-\nu\frac{\partial^2 u_0}{\partial y^2}=A\,\alpha\,t_1^{\alpha-1}\,W$$

$$u_0=0, \qquad y=0; \qquad u_0=U_d, \qquad y=\infty.$$

If the solution of this equation is sought in the form of

$$(6.2) \qquad u_0=At_1^{\alpha}\,W(x)\,\Phi_0'(\bar{\eta})$$

for the determination of the function $\Phi_0'(\bar{\eta})$, the substitution of (6.2) in (6.1) results in the differential equation

$$(6.3) \qquad \Phi_0'''+2\,\bar{\eta}\,\Phi_0''-4\,\alpha\,\Phi_0'=-4\,\alpha$$

where boundary conditions are

$$(6.4) \qquad \Phi_0(0)=\Phi_0'(0)=0, \quad \Phi_0'(\infty)=1.$$

The solution of (6.3) is given in terms of the gamma function $\Gamma(\alpha+1)$ and Gauss' function of error $g_{\alpha}(\bar{\eta})$:

$$(6.5) \qquad \Phi_0'(\bar{\eta})=1-2^{2\alpha}\,\Gamma(\alpha+1)\,g_{\alpha}(\bar{\eta})$$

$$(6.6) \qquad g_{\alpha}(\bar{\eta})=\frac{2}{\sqrt{\pi}\,\Gamma(2\,\alpha+1)}\int_{\bar{\eta}}^{\infty}(\gamma-\bar{\eta})^{2\alpha}\,e^{-\gamma^2}\,d\gamma$$

Since it is possible to obtain v_0 from the equation of continuity

$$v_0=-A\,t_1^{\alpha}W'\,2\sqrt{\nu\,t_1}\,\Phi_0(\bar{\eta})$$

no difficulties should be encountered in setting up an equation for the second approximation:

$$(6.7) \qquad \frac{\partial u_1}{\partial t_1}-\nu\frac{\partial^2 u_1}{\partial y^2}=A\,t_1^{\alpha}\,(UW'+U'W)\left(1-\Phi_0'\right)+ + A^2\,t_1^{2\alpha}\,WW'\left(1-\Phi_0'^2+\Phi_0\,\Phi_0''\right)$$

which is to be solved for the following boundary conditions:

$$u_1=0, \quad y=0; \qquad u_1=0, \quad y=\infty.$$

If the solution of the equation (6.7) is sought in the form of

$$(6.8) \qquad u_1=A\,t_1^{\alpha+1}\,(UW'+U'W)\,\Phi_1'(\bar{\eta})+A^2\,t_1^{2\alpha+1}\,WW'\,\Phi_2'(\bar{\eta}).$$

Then, from (6.7) there follows a differential equation

$$(6.9)\qquad \begin{cases} \Phi_1''' + 2\bar{\eta}\,\Phi_1'' - 4(\alpha+1)\,\Phi_1' = 4\left(\Phi_0' - 1\right) \\ \Phi_2''' + 2\bar{\eta}\,\Phi_2'' - 4(2\alpha+1)\,\Phi_2' = 4\left(\Phi_0'^2 - \Phi_0\,\Phi_0'' - 1\right) \end{cases}$$

with boundary conditions

$$(6.10)\qquad \begin{cases} \Phi_1(0) = \Phi_1'(0) = \Phi_1'(\infty) = 0 \\ \Phi_2(0) = \Phi_2'(0) = \Phi_2'(\infty) = 0. \end{cases}$$

The solutions of (6.9) fulfilling the conditions (6.10) are

$$\Phi_1'(\bar{\eta}) = -2^{2\alpha+2}\,\Gamma(\alpha+2)\,g_{\alpha+1}(\bar{\eta}) + 2^{2\alpha}\,\Gamma(\alpha+1)\,g_\alpha(\bar{\eta})$$

$$\Phi_2'(\bar{\eta}) = 2^{2\alpha+1}\,\Gamma(\alpha+1)\,\frac{1-\alpha}{1+\alpha}\,g_\alpha(\bar{\eta}) - 2^{2\alpha-1}\,\frac{\Gamma^2(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)}\,g_{\alpha-\frac{1}{2}}(\bar{\eta}) +$$

$$+\,2^{2\alpha-1}\,\frac{\Gamma(\alpha+1)}{\alpha+2}\,g_{\alpha-1}(\bar{\eta}) + 2^{4\alpha+1}\,\Gamma^2(\alpha+1)\left[g^2_{\alpha+\frac{1}{2}}(\bar{\eta}) - g_\alpha(\bar{\eta})\,g_{\alpha+1}(\bar{\eta})\right] -$$

$$-\,2^{4\alpha+2}\,\Gamma(2\alpha+2)\left[\frac{3-4\alpha}{2+2\alpha} + \frac{2\alpha}{\alpha+2} - \frac{\Gamma^2(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{5}{2}\right)} + \frac{1}{2}\,\frac{\Gamma^2(\alpha+1)}{\Gamma^2\left(\alpha+\frac{3}{2}\right)}\right] g_{2\alpha+1}(\bar{\eta})$$

Thus, we have the solution of the boundary layer for the duration of the additional power series accelerated motion:

$$(6.11)\qquad \begin{aligned} u = U f_1'(\eta) + A t_1^\alpha W\,\Phi_0'(\bar{\eta}) + A t_1^{\alpha+1}(UW' + U'W)\,\Phi_1'(\bar{\eta}) + \\ + A^2 t_1^{2\alpha+1}\,WW'\,\Phi_1'(\bar{\eta}). \end{aligned}$$

7. Power-series accelerated motion, following a preceding motion with constant acceleration. — If a body was moving with a constant acceleration $U_\gamma = tW(x)$, and thereupon was given an additional motion $U_d = A t_1^\alpha V(x)$ from (1.4) it follows

$$(7.1)\qquad \begin{cases} \dfrac{\partial u_0}{\partial t_1} - \nu\,\dfrac{\partial^2 u_0}{\partial y^2} = A\,\alpha\,t^{\alpha-1}\,V(x) \\ u_0 = 0,\quad y = 0; \qquad u_0 = U_d,\quad y = \infty. \end{cases}$$

If the solution of this equation is sought in the form of

$$u_0 = A\,t_1^\alpha\,V(x)\,\Phi_0'(\bar{\eta})$$

for the determination of the function $\Phi_0'(\bar{\eta})$, obviously we shall obtain an equation identical with (6.3) and thus its solution is (6.5).

The equation for the second approximation of the boundary layer velocity now becomes:

$$\frac{\partial u_1}{\partial t_1}-\nu\frac{\partial^2 u_1}{\partial y^2}=AT t_1^{\alpha}(W'V+WV')\left(1-\Phi_0'\right)+At_1^{\alpha}(W'V+ \tag{7.2}$$
$$+WV')\left(1-\Phi_0'\right)+A^2 t_1^{2\alpha}VV'\left(1-\Phi_0'^2+\Phi_0\Phi_0''\right).$$

Because of the initial condition $u_1=0$ for $t_1=0$, its solution should be sought as

$$u_1=AT t_1^{\alpha+1}(W'V+WV')\Phi_1'(\overline{\eta})+At_1^{\alpha+2}(W'V+ \tag{7.3}$$
$$+WV')\Phi_2'(\overline{\eta})+A^2 t_1^{2\alpha+1}VV'\Phi_3'(\overline{\eta}).$$

The substitution of (7.3) in (7.2) leads to a differential equation

$$\begin{cases} \Phi_1'''+2\overline{\eta}\,\Phi_1''-4(\alpha+1)\Phi_1'=-4\left(1-\Phi_0'\right) \\ \Phi_2'''+2\overline{\eta}\,\Phi_2''-4(\alpha+2)\Phi_2'=-4\left(1-\Phi_0'\right) \\ \Phi_3'''+2\overline{\eta}\,\Phi_3''-4(2\alpha+1)\Phi_3'=-4\left(1-\Phi_0'^2+\Phi_0\Phi_0''\right) \end{cases} \tag{7.4}$$

which have to be solved for the conditions

$$\begin{cases} \Phi_1(0)=\Phi_1'(0)=\Phi_1'(\infty)=0 \\ \Phi_2(0)=\Phi_2'(0)=\Phi_2'(\infty)=0 \\ \Phi_3(0)=\Phi_3'(0)=\Phi_3'(\infty)=0. \end{cases} \tag{7.5}$$

The solutions of (7.4) which fulfil the conditions (7.5) are

$$\Phi_1'(\overline{\eta})=2^{2\alpha}\Gamma(\alpha+1)g_\alpha(\overline{\eta})-2^{2\alpha+2}\Gamma(\alpha+2)g_{\alpha+1}(\overline{\eta})$$

$$\Phi_2'(\overline{\eta})=2^{2\alpha-1}\Gamma(\alpha+1)g_\alpha(\overline{\eta})-2^{2\alpha-1}\Gamma(\alpha+1)\frac{g_\alpha(0)}{g_{\alpha+2}(0)}g_{\alpha+2}(\overline{\eta})$$

$$\Phi_3'(\overline{\eta})=2^{2\alpha+1}\Gamma(\alpha+1)\frac{1-\alpha}{1+\alpha}g_\alpha(\overline{\eta})-2^{2\alpha-1}\frac{\Gamma^2(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)}g_{\alpha-\frac{1}{2}}(\overline{\eta})+$$

$$+2^{2\alpha-1}\frac{\Gamma(\alpha+1)}{\alpha+2}g_{\alpha-1}(\overline{\eta})+2^{4\alpha+1}\Gamma^2(\alpha+1)\left[g^2_{\alpha+\frac{1}{2}}(\overline{\eta})-g_\alpha(\overline{\eta})g_{\alpha+1}(\overline{\eta})\right]-$$

$$-2^{4\alpha+2}\Gamma(2\alpha+2)\left[\frac{3-4\alpha}{2+2\alpha}+\frac{2\alpha}{\alpha+2}-\frac{\Gamma^2(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{5}{2}\right)}+\right.$$

$$\left.+\frac{1}{2}\frac{\Gamma^2(\alpha+1)}{\Gamma^2\left(\alpha+\frac{3}{2}\right)}\right]g_{2\alpha+1}(\overline{\eta})$$

The addition of the preceding boundary layer velocity with expressions for „u_0" and „u_1" results in the total longitudinal velocity in the boundary layer

$$(7.6)\quad \begin{cases} u=(T+t_1)\,Wf_1'(\eta)+At_1^{\alpha}V(x)\,\Phi_0'(\overline{\eta})+ATt_1^{\alpha+1}(W'V+ \\ \quad +WV')\,\Phi_1'(\overline{\eta})+At_1^{\alpha+2}(WV'+W'V)\,\Phi_2'(\overline{\eta})+A^2t_1^{2\alpha+1}VV'\Phi_3'(\overline{\eta}). \end{cases}$$

The application of the equation of continuity readily yields the value of the velocity second projection „v".

8. Conclusion. — By this solution the additional boundary layer, or the boundary layer for the duration of the additional motion, acquired a property of an asymptotic boundary layer, i. e., it reaches the infinite distance from the body. Practically, however, we know that the boundary layer thickness is limited and that it is only several millimeters (*Example*: it is a well-known fact that the thickness of the boundary layer is a quantity of the order of magnitude $\frac{l}{\sqrt{Re}}=\sqrt{\frac{l\nu}{V}}$; if it is assumed that $l=1$ m, $V=1$ m/s, $\nu=0{,}01\frac{\text{cm}^2}{\text{s}}$ (water at 20° C), then $\sqrt{\frac{l\nu}{V}}=0{,}1$ cm $=1$ mm, and thus it is proved that the boundary layer thickness is really very small.)

The solution of an asymptotic boundary layer is of great importance because it involves the entire space around the body; practically, from $y=0{,}4$ cm to $y=\infty$ since a good agreement has been proved of this solution and a preceding one [1] at the same place, with a minimum discrepancy of 0.25% at most.

Of particular interest is the fact that this solution continues to give good results even in the immediate vicinity of the body down to the very contour of the body, negligibly disagreeing from the exact solution [1] for the same space.

It is not unrealistic to consider that this solution in the first approximation represents also the solution of the non-steady boundary layer on a body which was started from the state of certain preceding steady motions, even with the possibility of a certain rough differentiation consisting in either this preceding motion having been started impulsively, constantly accelerated or powerseries accelerated — by means of the "preceding" external potential velocity "U_s", which represents the preceding motion in the equations (1.4) and (1.5).

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