PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 8 (22), 1968, p. 23—36.

# LAMINAR BOUNDARY LAYER ON SLENDER BODIES OF REVOLUTION

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(Communicated May 25, 1966)

## NOMENCLATURE

$x$, $y$ — usual coordinates of the boundary layer,
$u$, $v$ — velocity projections in directions of axes $x$ and $y$,
$L$ — characteristic dimension of the body,
$r_0(x)$ — radius of the body cross section,
$U(x)$ — main stream velocity,
$\rho$, $\nu$, $\mu = \text{const}$ — usual symbols for the well known physical properties of the fluid,
$p$ — pressure,
$\tau_w = \mu (u_y)_{y=0}$ — skin friction,
$\delta(x)$ — boundary layer thickness,
$A_1(x)$ — displacement area and
$A_2(x)$ — momentum defect area.
The meaning of other symbols used will be given upon the introduction of such symbols in the paper.

## § 1 — INTRODUCTION

Problems of an axial flow past bodies of revolution can be divided into two groups. The first of these comprises problems in which the ratio between the boundary layer thickness and the radius of the body cross section can be considered much less than unity, while the other group comprises those problems in which the above ratio is approximately equal to unity, or even greater than unity. For the sake of brevity, the problems of the first group are called problems in which the transverse curvature effect has been neglected. It is a well known fact that in this case the boundary layer equations by means of *Mangler-Stepanov* transforms are reduced to those of two-dimensional problems. This fact made possible for problems of the first group to be solved by *Salnikov* [2] by means of *Görtler's* expansion [1] in a very general case of flow: a flow past solid bodies of revolution with the forward stagnation point and a flow past ring-like bodies of revolution of arbitrary shapes involving arbitrary distributions of the main stream velocities. The problems belonging to the second group are, also for the sake of brevity, called problems in which the transverse curvature effect has been taken into account. There exist so-called generalized transforms of *Probstein-Elliott* [3] by means of which in this case the boundary layer equations are reduced to those of the twodimensional problem, but only formally, since in them the kinematic viscosity $\bar{\nu}$ is not constant, but is the following function of the coordinates: $\bar{\nu} = \nu r^2/r_0^2$. Since, however, such a two-dimensional problem has not been solved, these transforms are of no practical importance. This was probably the reason for a relatively small number of papers dealing with this area of the boundary layer theory, such papers dealing with some individual special cases only. For instance, the problem of a flow past a semi-infinite circular cylinder in a region in which $\delta(x) < r_0(x)$ was dealt with by *Seban-Bond* [4] and *Kelly* [5], while the interval in which $\delta(x) > r_0(x)$ was dealt with by *Glauert-Lighthill* [6] and *Stewartson* [7]. Similar solutions of the basic equations, and some other special cases as well, have been dealt with in a paper by *Probstein-Elliott* [3] mentioned already in one of the earlier papers [8] of the present author.

The purpose of this paper is to solve the problems in which the transverse curvature effect is taken into consideration for a general case of flow as was done in [2] and in problems in which $\delta(x) \ll r_0(x)$. This puprose is achieved by introducing the so-called characteristic parameter which is proportional to the ratio of the boundary layer thickness and the radius of the body cross section, thus obtaining a solution for the interval in which this ratio is less than unity, in the form of a potential series in terms of a characteristic parameter, while the solution for the region in which this ratio is greater than unity is of the form of an asymptotic series, the specific form of which is given by a logarithmic behaviour of the velocity profile within the immediate vicinity of the body. Finally, a number of numerical examples is given.

## § 2 — SOLUTION OF THE PROBLEM WHEN $\delta(x) < r_0(x)$

The basic equations of the boundary layer for an axial flow past bodies of revolution for an arbitrarily given ratio of the boundary layer thickness and the radius of the body cross section are as follows:

$$
\begin{aligned}
& u u_x + v u_y = U U' + \nu \left( u_{yy} + \frac{1}{r_0 + y} u_y \right) \\
& [(r_0 + y) u]_x + [(r_0 + y) v]_y = 0
\end{aligned} \tag{1}
$$

with the following boundary conditions:

$$
\begin{aligned}
&\text{for} \quad y = 0 \qquad u = v = 0 \\
&\text{for} \quad y \to \infty \qquad u \to U(x).
\end{aligned}
$$

A transformation of these equations has been carried out already in a convenient way [8]. New variables

$$
\xi = \frac{1}{\nu L^2} \int_0^x U r_0^2 \, dx, \qquad \eta = \frac{U r_0 y}{\nu L \sqrt{2\xi}} \left(1 + \frac{y}{2 r_0}\right),
$$

$$
\psi(x, y) = \nu L \sqrt{2\xi} F(\xi, \eta)
$$

have been introduced, where the stream function $\psi(x, y)$ is defined by the following relations:

$$(r_0+y)\,u=\psi_y, \qquad (r_0+y)\,v=\psi_x.$$

Hence, for the dimensionless stream function $F(\xi, \eta)$ the following transformed equation of the boundary layer was obtained:

$$F_{\eta\eta\eta}+FF_{\eta\eta}+\beta(\xi)\left(1-F_\eta^2\right)=2\,\xi\,(F_\eta F_{\xi\eta}-F_\xi F_{\eta\eta})-\Delta(\xi)\,(\eta F_{\eta\eta\eta}+F_{\eta\eta}) \tag{2}$$

for which the boundary conditions were

$$\text{for} \quad \eta=0 \qquad F=F_\eta=0$$
$$\text{for} \quad \eta\to\infty \qquad F_\eta\to 1.$$

In this equation we have

$$\beta(\xi)=\frac{2\,\xi\nu L^2 U'}{U^2 r_0^2}, \qquad \Delta(\xi)=\frac{2\,\nu L\sqrt{2\,\xi}}{U r_0^2}$$

where $\beta(\xi)$ is the so-called principal function which is well known [2], while $\Delta(\xi)$ is a quantity which it was shown [8] is, in a most general case of flow, proportional to the ratio $\delta(x)/r_0(x)$, this being the reason for calling this quantity a characteristic parameter.

For those values of the variable $\xi$ for which $\Delta(\xi)<1$, the solution of the equation (2) was sought in the form of a series:

$$F(\xi, \eta)=F_0(\xi, \eta)+F_1(\xi, \eta)\,\Delta(\xi)+F_2(\xi, \eta)\,\Delta^2(\xi)+\cdots \tag{3}$$

where for the first term $F_0(\xi, \eta)$, which gives an exact solution for the case when the transverse curvature effect can be neglected, we obtained the well known *Görtler's* equation [1]. The equation defining the second term of the series used is linear and is given as follows:

$$\begin{aligned} &F_{1\eta\eta\eta}+(F_0+2\,\xi\,F_{0\xi})\,F_{1\eta\eta}-2\,\xi\,F_{0\eta}\,F_{1\xi\eta}-\{2\,\xi\,F_{0\xi\eta}+[2\,\beta(\xi)+\gamma(\xi)]\,F_{0\eta}\}\,F_{1\eta}+\\ &\quad+2\,\xi\,F_{0\eta\eta}\,F_{1\xi}+[1+\gamma(\xi)]\,F_{0\eta\eta}\,F_1=-(\eta\,F_{0\eta\eta\eta}+F_{0\eta\eta}) \end{aligned} \tag{4}$$

with the boundary conditions:

$$\text{for} \quad \eta=0 \qquad F_1=F_{1\eta}=0$$
$$\text{for} \quad \eta\to\infty \qquad F_{1\eta}\to 0,$$

In this equation, in addition to $\beta(\xi)$, a new principal function $\gamma(\xi)$ occurs, and is defined by:

$$\gamma(\xi)=\frac{2\,\xi\,\Delta'(\xi)}{\Delta(\xi)}=1-\alpha(\xi)\,\beta(\xi)$$

where $\alpha(\xi)$ may be called an auxiliary principal function:

$$\alpha(\xi)=1+2\,\frac{U r_0'}{U' r_0}.$$

In the paper [8] already mentioned, similar solutions of (2) were considered as well as those simple solutions in which the coefficients of the series (3) are functions of the variable $\eta$ only. Here, we propose to discuss a very general case.

It is well known that *Görtler's* equation for the first term of the series (3) has been solved by expansion into a series of the principal function $\beta(\xi)$ [2] and by introducing a system of universal functions [1] which were such that it was possible to tabulate them once and for all. In order to obtain a solution for the equation (4) in the same manner, it is necessary to consider the possibility that the new principal function $\gamma(\xi)$ and the auxiliary principal function $\alpha(\xi)$, can be expanded into a series of the same form as was the case with the series used for the function $\beta(\xi)$. If the expressions for $U(x)$ and $r_0(x)$ [2] are taken into account, then it is possible to show that in all rotationally symmetrical problems important in practice, the new principal function $\gamma(\xi)$ can be expanded into a series of the following form:

$$\gamma(\xi)=\sum_{k=0}^{\infty}\gamma_{nk}\,\xi^{nk}$$

where $n=1$ and $\gamma_0=1$, for ring-like bodies of revolution with an acute leading edge, and $n=1/2$ and $\gamma_0=-1/2$, and $\gamma_0=0$ for solid bodies of revolution with a forward stagnation point and ring-like wings, respectively.

Now, the solution of (4) can be sought in form of a series:

$$F_1(\xi,\eta)=\sum_{k=0}^{\infty}F_{1\,nk}(\eta)\,\xi^{nk} \tag{5}$$

so that for the first term we shall obtain the following equation:

$$F_{10}'''+F_{00}F_{10}''-(2\beta_0+\gamma_0)F_{00}'F_{10}'+(1+\gamma_0)F_{00}''F_{10}=-\left(\eta F_{00}'''+F_{00}''\right) \tag{6}$$

with the boundary conditions:

$$F_{10}(0)=F_{10}'(0)=F_{10}'(\infty)=0.$$

. For the remaining coefficients-functions of the series (5) we shall obtain a recursive system of ordinary, linear differential equations of the third order into which it will be possible to introduce instead of coefficients-functions, universal functions $p\ldots(\eta)$, $q\ldots(\eta)$, and $t\ldots(\eta)$, by means of the following linear combinations:

$$F_{11}=\beta_1p_1+\gamma_1q_1$$
$$F_{12}=\beta_1^2p_{11}+\beta_2p_2+\beta_1\gamma_1t_{1,1}+\gamma_1^2q_{11}+\gamma_2q_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

when $n=1$, and:

$$F_{1\frac{1}{2}}=\beta_{\frac{1}{2}}p_{\frac{1}{2}}+\gamma_{\frac{1}{2}}q_{\frac{1}{2}}$$
$$F_{11}=\beta_{\frac{1}{2}}p_{\frac{1}{2}\frac{1}{2}}+\beta_1p_1+\beta_{\frac{1}{2}}\gamma_{\frac{1}{2}}t_{\frac{1}{2},\frac{1}{2}}+\gamma_{\frac{1}{2}}^2q_{\frac{1}{2}\frac{1}{2}}+\gamma_1q_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

when $n=1/2$.

The boundary conditions will be:

$$p\ldots(0)=p'\ldots(0)=p'\ldots(\infty)=0$$
$$q\ldots(0)=q'\ldots(0)=q'\ldots(\infty)=0$$
$$t\ldots(0)=t'\ldots(0)=t'\ldots(\infty)=0,$$

The equations themselves for the universal functions as well as the expressions for the remaining coefficients of the series used for the new principal function will not be shown here because of lack of space.

It is possible to derive and to solve in the same way the equations for remaining coefficients of the series (3). If only the first two terms are taken into account, then the expressions for the skin friction and for the displacement area will be:

$$\frac{r_0(x)\,\tau_w(x)}{2\,\mu\,U(x)} = \frac{1}{\Delta(\xi)}\,F_{0\eta\eta}(\xi,0) + F_{1\eta\eta}(\xi,0)$$

$$\frac{A_1}{\pi r_0^2} = \Delta(\xi)\,[\eta_0(\xi) - \Delta(\xi)\,\eta_1(\xi)]$$

where:

$$\eta_0(\xi) = \lim_{\eta\to\infty}[\eta - F_0(\xi,\eta)], \quad \eta_1(\xi) = \lim_{\eta\to\infty} F_1(\xi,\eta)$$

## § 3 — SOLUTION OF THE PROBLEM FOR THE CASE WHEN $\delta(x) > r_0(x)$

If for some values of $x$, i. e. $\xi$, $\Delta(\xi) > 1$, then the solution of the equation (2) cannot be sought in the form of the series (3), but we must seek a solution in the form of certain asymptotic series in terms of the characteristic parameter $\Delta(\xi)$. It is evident that we shall be able, regardless of the form of the asymptotic series involved, to assume for the first, i. e. the leading, term $F_0(\xi,\eta)$ than it represents a function of $\eta$ only, and thus obtain the following differential equation:

(7) $$\eta F_0''' + F_0'' = 0$$

for which the boundary conditions are:

$$F_0(0) = F_0'(0) = 0, \qquad F_0'(\infty) = 1.$$

The general solution of this equation is:

$$F_0'(\eta) = C + D \ln \eta$$

where $C$ and $D$ are the constants of integration. Hence, it is clear that these boundary conditions cannot be fulflled. Therefore, the equation (7) cannot be solved with appropriate universally formulated boundary conditions, and the solution of (2) cannot be sought in the form of any asymptotic series in terms of the characteristic parameter. The reason for non-existence of the solution in this form is the universal formulation of the boundary conditions, which brings forth the unlimitedness of solution both on the surface of the body and in the infinity.

Therefore, in case when $\Delta(\xi) > 1$, the transformation of the basic equations of the boundary layer will be carried out in an entirely different way, in which instead of the variable $y$, use will be made of the variable $r = r_0(x) + y$, which leads to a non-universal formulation of the internal boundary condition. If the basic equations (1) are written in terms of variables $x$ and $r$, we shall obtain:

$$uu_x + vu_r = UU' + \nu\left(u_{rr} + \frac{1}{r}\,u_r\right)$$

$$(ru)_x + (rv)_r = 0$$

with the boundary conditions:

$$\text{for} \quad r = r_0(x) \qquad u = v = 0$$

$$\text{for} \quad r \to \infty \qquad u \to U(x).$$

Now, we shall transform these equations by introducing new variables $\xi$ and $\varphi$, and a new dimensionless stream function $W(\xi, \varphi)$ in the following manner:

$$\xi = \frac{1}{\nu L^2} \int_0^x U r_0^2 dx, \qquad \varphi = \frac{1}{\Delta^2(\xi)} \frac{r^2}{r_0^2},$$

$$\psi(x, r) = \frac{\Delta^2(\xi)}{2} U r_0^2 W(\xi, \varphi).$$

Hence, the transformed equation of the boundary layer will be:

$$\varphi W_{\varphi\varphi\varphi} + \{1 + [1 + \gamma(\xi)] W\} W_{\varphi\varphi} + \beta(\xi)(1 - W_\varphi^2) = \tag{8}$$
$$= 2\xi (W_\varphi W_{\xi\varphi} - W_\xi W_{\varphi\varphi})$$

with the non-universally formulated boundary conditions:

$$\text{for} \quad \varphi = \frac{1}{\Delta^2(\xi)} \qquad W = W_\varphi = 0$$

$$\text{for} \quad \varphi \to \infty \qquad W_\varphi \to 1.$$

The solution of this equation will be sought in the form of an asymptotic series in terms of what for the time being will be an arbitrary function of the characteristic parameter $f(\Delta)$:

$$W(\xi, \varphi) = W_0(\xi, \varphi) + \frac{W_1(\xi, \varphi)}{f(\Delta)} + \frac{W_2(\xi, \varphi)}{f^2(\Delta)} + \dots \tag{9}$$

For the coefficients of this series we shall obtain a recursive system of differential equations involving principal functions $\beta(\xi)$ and $\gamma(\xi)$. The solutions of these functions should be then sought in form of series in terms of $\xi$, with coefficients-functions of $\varphi$, while the form of these series would depend upon the form of the corresponding series used for the principal function. It is little probable that with ring-like bodies of revolution $\Delta(\xi) > 1$ for a certain $\xi$, therefore, in future we shall restrict ourselves only to solid bodies of revolution. With the latter (§ 2), $n = 1/2$, $\beta_0 = 1/2$, and $\gamma_0 = -1/2$; thus, the series for the coefficients of the series (9) will have the following form:

$$W_i(\xi, \varphi) = \sum_{j=0}^{\infty} W_{i\frac{j}{2}}(\varphi)\, \xi^{\frac{j}{2}}, \qquad i = 0, 1, 2, \dots \tag{10}$$

while the expression for the longitudinal projection of velocity will be:

$$\frac{u}{U} = W_\varphi = \sum_{i,j=0}^{\infty} \frac{W'_{i\frac{j}{2}}(\varphi)\, \xi^{\frac{j}{2}}}{f^i(\Delta)}. \tag{11}$$

When solving the problem of a flow past a circular cylinder, *Glauert-Lighthill* [6], as well as *Stewartson* [7], have found that the longitudinal projection of velocity in the vicinity of the surface of the cylinder is proportional to the logarithm of the distance from the cylinder axis. Now, we shall assume here that the longitudinal projection of velocity is proportional to the logarithm of distance from the axis of a body also in case of a flow past a body of an arbitrary shape. If this assumption is justified, it will mean that each of the coefficients-functions of the series (10) for small values of the variable $\varphi$ will have to behave in the following manner:

$$W'_{i\frac{j}{2}}(\varphi) \sim C_{i\frac{j}{2}} + D_{i\frac{j}{2}} \ln\varphi \qquad (i, j = 0, 1, 2, \dots)$$

where $C_{i\frac{j}{2}}$ and $D_{i\frac{j}{2}}$ are arbitrary constants. On the surface of the body itself, that is for $\varphi = 1/\Delta^2$, we shall have:

$$W'_{i\frac{j}{2}}(1/\Delta^2) = C_{i\frac{j}{2}} - 2\,D_{i\frac{j}{2}} \ln\Delta. \tag{12}$$

If now it is wished to fulfil the internal boundary condition: $W_\varphi = 0$, for $\varphi = 1/\Delta^2$, and thence to obtain the equations for determination of constants $C_{i\frac{j}{2}}$ and $D_{i\frac{j}{2}}$, then it is obvious that the up-till-now arbitrary function $f(\Delta)$ must have the following form: $f(\Delta) = \ln\Delta$. The fulfilment of the above boundary conditions results in:

$$\sum_{i,\,j=0}^{\infty} \frac{\left(C_{i\frac{j}{2}} - 2\,D_{i\frac{j}{2}} \ln\Delta\right)\xi^{\frac{j}{2}}}{\ln^i\Delta} = 0$$

whence by comparison of coefficients of terms of the same order of $\ln\Delta$ we obtain:

$$-2\,D_{00} - 2\,D_{0\frac{1}{2}}\,\varphi^{\frac{1}{2}} - 2\,D_{01}\,\xi \cdots = 0$$

$$\sum_{j=0}^{\infty}\left(C_{i\frac{j}{2}} - 2\,D_{i+1,\,\frac{j}{2}}\right)\xi^{\frac{j}{2}} = 0, \quad i = 0,\ 1,\ 2, \dots$$

Since the equations thus obtained must be satisfied for every $\xi$, it is obvious that we must have:

$$\begin{aligned} &D_{0\frac{j}{2}} = 0 \\ &C_{i\frac{j}{2}} = 2\,D_{i+1,\,\frac{j}{2}} \end{aligned} \qquad (i,\ j = 0,\ 1,\ 2, \dots). \tag{13}$$

From the transformed equation (8) of the boundary layer the following equations for the first two coefficients of the series (9) can be obtained:

$$\varphi\,W_{0\varphi\varphi\varphi} + \{1 + [1 + \gamma)\xi)]\,W_0\}\,W_{0\varphi\varphi} + \beta(\xi)\left(1 - W_{0\varphi}^2\right) = 2\,\xi\,(W_{0\varphi}\,W_{0\xi\varphi} - W_{0\xi}\,W_{0\varphi\varphi}) \tag{14}$$

$$\begin{aligned} &\varphi\,W_{1\varphi\varphi\varphi} + \{1 + 2\,\xi\,W_{0\xi} + [1 + \gamma(\xi)]\,W_0\}\,W_{1\varphi\varphi} - 2\,\xi\,W_{0\varphi}\,W_{1\xi\varphi} - \\ &-2\,[\beta(\xi)\,W_{0\varphi} + \xi\,W_{0\xi\varphi}]\,W_{1\varphi} + 2\,\xi\,W_{0\varphi\varphi}\,W_{1\xi} + [1 + \gamma(\xi)]\,W_{0\varphi\varphi}\,W_1 = 0. \end{aligned} \tag{15}$$

In view of the fact that the internal boundary layer for $W_\varphi$ has already been fulfilled by the equations (13), the remaining conditions for coefficients $W_0(\xi, \varphi)$ and $W_1(\xi, \varphi)$, will be:

$$\text{for} \quad \varphi = 1/\Delta^2 \qquad W_0 = W_1 = 0$$

$$\text{for} \quad \varphi \to \infty \qquad W_{0\varphi} \to 1, \ W_{1\varphi} \to 0$$

the first of which can be replaced by an approximate value:

$$\text{for} \quad \varphi \to 0 \qquad W_0 \to 0, \ W_1 \to 0$$

because here, in this way for large values of the characteristic parameter, which are dealt with at present, we make a smaller error than if we take even an arbitrary large number of terms of the series (9).

From (14), we shall obtain for the first terms of the series (10), when $i=0$, as follows:

$$\varphi W_{00}''' + \left(1 + \frac{1}{2} W_{00}\right) W_{00}'' + \frac{1}{2}\left(1 - W_{00}'^2\right) = 0 \tag{16}$$

with the boundary conditions:

$$\text{for} \quad \varphi \to 0 \qquad W_{00} \to 0, \ W_{00}' \sim C_{00}$$

$$\text{for} \quad \varphi \to \infty \qquad W_{00}' \to 1$$

where $C_{00}$ is an arbitrary constant. The solution of this equation in the vicinity of the point $\varphi = 0$ will be sought in the form of the following power series:

$$W_{00}(\varphi) = \sum_{k=0}^{\infty} h_k \varphi^k .$$

By substitution in the equation mentioned above and by applying the first two boundary conditions, we shall have:

$$h_0 = 0, \ h_1 = C_{00}, \ h_2 = (C_{00}^2 - 1)/4, \ h_3 = C_{00} h_2/12, \ \ldots$$

It should be noted that for a definite $C_{00}$, it is possible to work out all the coefficients of the series regardless of the fact whether the boundary condition is fulfilled or not! It cannot be expected, of course, that this boundary condition will be automatically fulfilled for any $C_{00}$; thus, the conclusion is naturally reached that (16) has no solution for any value of $C_{00}$. Therefore, it is obvious that the boundary condition will be fulfilled in the infinity if $C_{00} = 1$. All coefficients $h_k$ will then vanish for $k \geqslant 2$, and the solution of (16) will be very simple, indeed:

$$W_{00}(\varphi) = \varphi .$$

Now, the equation for the second term of the series (10) for $i = 0$ will be:

$$\varphi W_{0\frac{1}{2}}''' + \left(1 + \frac{\varphi}{2}\right) W_{0\frac{1}{2}}'' - 2 W_{0\frac{1}{2}}' = 0 \tag{17}$$

with the boundary conditions:

$$\text{for} \quad \varphi \to 0 \qquad W_{0\frac{1}{2}} \to 0, \ W_{0\frac{1}{2}}' \sim C_{0\frac{1}{2}}$$

$$\text{for} \quad \varphi \to \infty \qquad W_{0\frac{1}{2}}' \to 0.$$

The general solution of this equation is:

$$W'_{0\frac{1}{2}}(\varphi) = M_{0\frac{1}{2}}\, e^{-\varphi/2}\, \Phi(5,\ 1;\ \varphi/2) + N_{0\frac{1}{2}}\, e^{-\varphi/2}\, G(5,\ 1;\ \varphi/2)$$

where $\Phi(5, 1; \varphi/2)$ and $G(5, 1; \varphi/2)$ are confluent hypergeometric functions of the first and second kind, respectively. If we take into account the asymptotic behaviour of these functions for $\varphi \to 0$ and $\varphi \to \infty$, then the application of the external boundary condition will result in $M_{0\frac{1}{2}} = 0$, while the application of the internal boundary condition will result in:

$$-\frac{N_{0\frac{1}{2}}}{\Gamma(5)}[\psi(5) - 2\psi(1) + \ln(\varphi/2)] \sim C_{0\frac{1}{2}}$$

where $\Gamma(x)$ are the so-called gamma functions, and $\psi(x)$ the logarithmic derivative of the gamma function. It is obvious tha we must have simultaneously: $N_{0\frac{1}{2}} = 0$ and $C_{0\frac{1}{2}} = 0$; thus, the equation (17) has only a trivial solution:

$$W_{0\frac{1}{2}}(\varphi) \equiv 0.$$

It can be shown that equations for other coefficients of the series (10) for $i = 0$, too, have only trivial solutions. Therefore, the first term of the series (9) is reduced to:

$$W_0(\xi,\ \varphi) = W_{00}(\varphi) = \varphi. \tag{18}$$

Also, we shall have: $C_{0\frac{j}{2}} = 0$, or, if (13) is taken into consideration: $D_{1\frac{j}{2}} = 0$, $j = 1, 2, 3, \ldots$ Since $C_{00} = 1$, $D_{10} = 1/2$.

If (18) is taken into account, then the equation (15) used for the evaluation of $W_1(\xi, \varphi)$ is greatly simplified:

$$\varphi W_{1\varphi\varphi\varphi} + \{1 + [1 + \gamma(\xi)]\varphi\} W_{1\varphi\varphi} - 2\xi W_{1\xi\varphi} - 2\beta(\xi) W_{1\varphi} = 0. \tag{19}$$

If the solution is assumed to be in the form of the series (10) with $i = 1$, then for the first term of the series we shall have the following equation:

$$\varphi W'''_{10} + \left(1 + \frac{\varphi}{2}\right) W''_{10} - W'_{10} = 0 \tag{20}$$

with the boundary conditions:

$$\text{for}\quad \varphi \to 0 \qquad W_{10} \to,\ W'_{10} \sim C_{10} + \frac{1}{2}\ln\varphi$$

$$\text{for}\quad \varphi \to \infty \qquad W'_{10} \to 0.$$

The general solution is:

$$W'_{10} = M_{10}\, e^{-\varphi/2}\, \Phi(3,\ 1;\ \varphi/2) + N_{10}\, e^{-\varphi/2}\, G(3,\ 1;\ \varphi/2)$$

where due to the external boundary solution, we must have $M_{10}=0$, while the internal boundary condition will result in:

$$-\frac{N_{10}}{\Gamma(3)}[\psi(3)-2\psi(1)+\ln(\varphi/2)]\sim C_{10}+\frac{1}{2}\ln\varphi$$

whence by equation the free terms and those with $\ln\varphi$ and by taking into consideration the values for the corresponding gamma functions and their logarithmic derivatives we have:

$$N_{10}=-1, \qquad C_{10}=\frac{1}{4}(3+2\gamma-2\ln 2)$$

where $\gamma=0.5772\ldots$ is Euler's constant.

Hence the solution of the equation (20) will finally be:

$$W_{10}(\varphi)=-\int_0^{\varphi} e^{-\varphi/2}G(3,\ 1;\ \varphi/2)\,d\varphi. \tag{21}$$

Now, we can set up an equation for the determination of coefficients $W_{1\frac{1}{2}}(\varphi)$:

$$\varphi W'''_{1\frac{1}{2}}\left(1+\frac{\varphi}{2}\right)W''_{1\frac{1}{2}}-2W'_{1\frac{1}{2}}=2\beta_{\frac{1}{2}}W'_{10}-\gamma_{\frac{1}{2}}\varphi W''_{10}$$

or, if we take into account (21), the rules for derivation of the confluent hypergeometric functions and certain recurrence relations fulfilled by them ([9], page 507):

$$\varphi W'''_{1\frac{1}{2}}+\left(1+\frac{\varphi}{2}\right)W''_{1\frac{1}{2}}-2W'_{1\frac{1}{2}}=-2\left(\beta_{\frac{1}{2}}-\gamma_{\frac{1}{2}}\right)e^{-\varphi/2}G(3,\ 1;\ \varphi/2)-$$
$$-\gamma_{\frac{1}{2}}e^{-\varphi/2}G(2,\ 1;\ \varphi/2)$$

with boundary conditions:

for $\varphi\to 0$ $\qquad W_{1\frac{1}{2}}\to 0,\ W'_{1\frac{1}{2}}\sim C_{1\frac{1}{2}}$

for $\varphi\to\infty$ $\qquad W'_{1\frac{1}{2}}\to 0.$

The general solution of this equation is:

$$W'_{1\frac{1}{2}}(\varphi)=M_{1\frac{1}{2}}e^{-\varphi/2}\Phi(5,\ 1;\ \varphi/2)+N_{1\frac{1}{2}}e^{-\varphi/2}G(5,\ 1;\ \varphi/2)+$$
$$+2\left(\beta_{\frac{1}{2}}-\gamma_{\frac{1}{2}}\right)e^{-\varphi/2}G(3,\ 1;\ \varphi/2)+\frac{2}{3}\gamma_{\frac{1}{2}}e^{-\varphi/2}G(2,\ 1;\ \varphi/2).$$

The constants $M_{1\frac{1}{2}}$ and $N_{1\frac{1}{2}}$ can be evaluated in the same way as those in case of the equation (20). We shall obtain:

$$M_{1\frac{1}{2}}=0, \quad N_{1\frac{1}{2}}=-8\left(3\beta_{\frac{1}{2}}-\gamma_{\frac{1}{2}}\right)$$

where must be:

$$C_{1\frac{1}{2}}=\frac{1}{36}\left(21\,\beta_{\frac{1}{2}}+5\,\gamma_{\frac{1}{2}}\right)$$

In exactly the same way, it is possible to derive and to solve in a closed form the equations for the remaining coefficients of the series (10) when $i=1$, and also when $i=2, 3, \ldots$ The calculations, however, rapidly become very complicated, and the use of an electronic computer is greatly warranted for practical reasons.

The expression for the skin friction becomes:

$$\frac{r_0(x)\,\tau_w(x)}{2\,\mu\,U(x)}=\frac{1}{\ln\Delta}\left(D_{10}+\frac{D_{20}+D_{2\frac{1}{2}}\,\xi^{\frac{1}{2}}+D_{21}\,\xi+\cdots}{\ln\Delta}+\cdots\right)$$

where:

$$D_{10}=1/2,\quad D_{20}=C_{10}/2,\quad D_{2\frac{1}{2}}=C_{1\frac{1}{2}}/2,\ \ldots$$

The first two terms of the series (9) will give the same value for both the displacement area and the momentum defect area. This value is:

$$\frac{A_1}{\pi r_0^2}=\frac{A_2}{\pi r_2^0}=-\frac{\Delta^2(\xi)}{\ln\Delta(\xi)}\,W_1(\xi,\ \infty).$$

Thus, a solution is obtained of the basic equations of the boundary layer for the case of great values of the characteristic parameter $\Delta(\xi)$ for an arbitrary shape of the body and an arbitrary distribution of the main stream velocity. The assumption on the logarithmic behaviour of the velocity profile in the vicinity of the body in such a general case of a flow is shown, therefore, to be quite justified.

## § 4 — NUMERICAL EXAMPLES

Now, we propose to show some numerical results for cases in which the difficulties of numerical nature are the least. Those are certainly the cases in which the coefficients of the series (9), as well as those of the series (3) are reduced to being functions only of the variables $\varphi$ or $\eta$, respectively. In the author's paper [8] already mentioned it was shown that in this case it is necessary to have:

$$\beta(\xi)=\beta_0=\text{const.}\quad\text{and}\quad\gamma(\xi)=\gamma_0=\text{const.}$$

and that these conditions will be fulfilled when:

$$U(x)=cx^m,\qquad r_0(x)=ax^n\quad(a,\ c>0).\tag{22}$$

The constants $\beta_0$ and $\gamma_0$, then have the following values:

$$\beta_0=\frac{2m}{m+2n+1},\qquad\gamma_0=-\frac{m+2n-1}{m+2n+1}.\tag{23}$$

In this case, the equation for the first term of the series (3) is reduced to the well known *Falkner-Skan* [10] equation, while (4) is reduced to (6) which is numerically integrated for values $\beta_0=0.18$; $-0.14$; 0 and 0.5, when $n=0$ (a circular cylinder), or when $n=0.5$ (a body having approximately the form of a paraboloid of revolution) and when $n=1$ (a cone). Thus, we have analyzed the transverse curvature effect of the body in an interval in which $\Delta(\xi)<1$, and now we proceed to obtain the corresponding numerical results for the interval $\Delta(\xi)>1$.

If the main stream velocity and the radius of the body cross section are given in the form of (22), then the series (9) will become:

$$W(\xi,\ \varphi)=W_0(\varphi)+\frac{W_1(\varphi)}{\ln\Delta}+\cdots$$

where we shall have $W_0(\varphi)=\varphi$, while $W_1(\varphi)$ will satisfy the equation:

$$\varphi W_1''' + [1+(1+\gamma_0)\varphi] W_1'' - 2\beta_0 W_1' = 0 \tag{24}$$

with the boundary conditions:

$$\text{for}\quad \varphi\to 0 \qquad W_1\to 0,\ W_1'\sim C_1+\frac{1}{2}\ln\varphi$$

$$\text{for}\quad \varphi\to\infty \qquad W_1'\to 0$$

this equation being obtainable from (19) provided we take into account that $W_{1\xi}\equiv 0$.

That substitution of variables:

$$(1+\gamma_0)\varphi=\zeta \quad \text{and} \quad W_1'(\varphi)=e^{-\zeta}\Phi(\zeta)$$

will result in the equation (24) being reduced to a confluent hypergeometric equation:

$$\zeta\Phi''+(1-\zeta)\Phi'-(2m+1)\Phi=0$$

the general solution of which is:

$$\Phi(\zeta)=M\,\Phi(2m+1,\ 1;\ \zeta)+NG(2m+1,\ 1;\ \zeta)$$

when $m\neq -1/2,\ -1,\ -3/2,\ \ldots$

It has already been shown [8] that practically only values $m>-1/2$ are to be considered, so it can be taken that the solution is valid for every $m$. The general solution of the equation (24) will be:

$$W_1'(\varphi)=Me^{-\zeta}\Phi(2m+1,\ 1;\ \zeta)+Ne^{-\zeta}G(2m+1,\ 1;\ \zeta)$$

where;

$$\zeta=\frac{2}{m+2n+1}\varphi.$$

The application of the external boundary condition results in the fact that there must be $M=0$, while the internal boundary condition will produce:

$$N=-\frac{\Gamma(2m+1)}{2}$$

$$C_1=\frac{1}{2}\left[\psi(2m+1)+2\gamma+\ln\frac{2}{m+2n+1}\right]$$

thus, the final solution of (24) will be:

$$W_1(\varphi) = -\frac{m+2n+1}{4}\,\Gamma(2m+1)\int\limits_0^{\frac{2}{m+2n+1}\varphi} e^{-\zeta}\, G(2m+, 1; \zeta)\, d\zeta.$$

If this solution is known, then the expressions for the skin friction and the characteristic surfaces of the boundary layer will be:

$$\frac{r_0(x)\,\tau_w(x)}{2\,\mu\,U(x)} = \frac{1}{\ln\Delta(\xi)}\left(D_1 + \frac{D_2}{\ln\Delta(\xi)}\right)$$

$$\frac{A_1}{\pi r_0^2} = \frac{A_2}{\pi r_0^2} = -\frac{\Delta^2(\xi)}{\ln\Delta(\xi)}\,W_1(\infty)$$

where: $D_1 = 1/2$ and $D_2 = C_1/2$.

In a special case considered by *Glauert-Lighthill* [6], $m = n = 0$ and, hence the expression for the skin friction becomes:

$$\frac{a\,\tau_w}{\mu\,c} = \frac{2}{\ln\dfrac{8\,\nu\,x}{ca^2}}\left(1 + \frac{\gamma + \ln 2}{\ln\dfrac{8\,\nu\,x}{ca^2}} + \cdots\right).$$

If this expression is compared with the corresponding expression obtained by *Glauert-Lighthill*, it will be noted that there exists a slight difference, but the numerical differences are negligible.

The values $C_1$ and $W_1(\infty)$ in cases considered are shown in Table 1.

Table 1

| | $\beta_0 = 0.5$ | | $\beta_0 = 0$ | | $\beta_0 = -0.14$ | | $\beta_0 = -0.18$ | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $C_1$ | $-W_1(\infty)$ | $C_1$ | $-W_1(\infty)$ | $C_1$ | $-W_1(\infty)$ | $C_1$ | $-W_1(\infty)$ |
| 0 | 0.870 | 0.202 | 0.635 | 0.251 | 0.489 | 0.263 | 0.437 | 0.265 |
| 0.5 | 0.742 | 0.312 | 0.289 | 0.503 | −0.038 | 0.575 | −0.139 | 0.624 |
| 1 | 0.691 | 0.338 | 0.086 | 0.755 | −0.435 | 1.119 | −0.562 | 1.373 |

On the grounds of these values, we have shown in Fig. 1 the diagrams of the resistance per unit length of the body, $F_l = 2\,\pi\,r_0(x)\,\tau_w(x)$ and of the displacement area for the case of flow past a circular cylinder ($n = 0$), when $\beta_0 = 0$ and $\beta_0 = -0.18$.

The solid curve shows the solutions obtained by series (3) and (9). The convergency of both these series, of course, stops in an interval in which $\Delta(\xi) \approx 1$ ($\lg\Delta(\xi) \approx 0$). The solution for this interval was obtained by interpolation (— · — · — · —). The broken curve represents the deviation which would arise if in this interval we use the results obtained by the series (3) and (9), while the curve shown in dots ....... represents results which would be obtained by taking into account only the first term of the series (3) which, as we have already pointed out, satifies the *Falkner-Skan* [10] equation and corresponds to the problem in which the transverse curvature effect has been

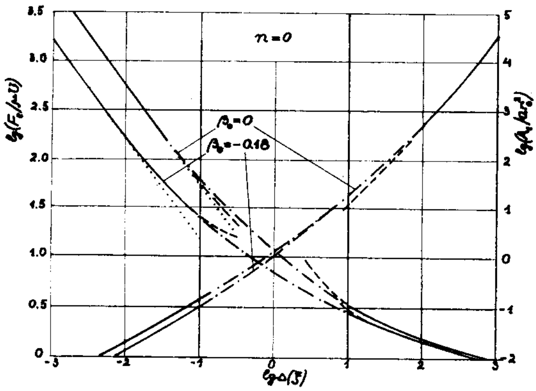


Fig. 1

neglected. In such a logarithmic diagram this solution is a straight line and it is quite clear that this line greatly departs in the intervals $\Delta(\xi) \approx 1$ and $\Delta(\xi) > 1$, as was to be expected, from the solution obtained in this paper.

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