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ON A-TREES

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1. Introduction.

1.1. At several opportunities (cf. the bibliography at the end of the article; in particular see my memory [8] written in 1937) one was lead to consider trees $A = A_\nu$ having the following properties:

1. *The height* γA *of* A *is an initial ordinal number* ω_ν;

2. *Every row* $R_\xi A := \{x;\ x \in A,\ \text{type of } A(\cdot, x) = \xi\}$ *is of a cardinality* $< k\,\omega_\nu$ *and* $\sup_\xi kR_\xi A = k\,\omega_{\nu^-}$ *(for an ordinal* ν *we define* ν^- *to be* $\nu - 1$ *or* ν, *according as* $\nu - 1$ *exists or does not exist; in particular* $0^- = 0$).

3. *The cardinality of every chain in* A *is* $< kA$.

4. *For every* $x \in A$ *the height* $\gamma[a]$ *of all the points of* A *each of which is comparable to* a *equals* γA.

5. *For every* $x \in A$ *the node* $|x|_A := \{x;\ x \in A,\ A(\cdot, x) = A(\cdot, a)\}$ *has* 1 *or* $k\gamma x$ *points, according as* $(\gamma x)^- = \gamma x$ *or* $(\gamma x)^- < \gamma x$; *here* γx *is the ordinal satisfying* $x \in R_{\gamma x} A$.

1.2. If ω_ν is cofinal to ω_0, A_ν does not exist; A_1 does exist (N. Aronszajn, v. [5], p. 96) as well as $A_{\alpha+1}$ for every ordinal α; in this paper we shall prove it without continuum hypothesis; the proof is analogous as it was done for the case $\alpha = 0$ in Đ. Kurepa [8] and is based purely on order considerations (cf. also [1], [15]).

The problem of the existence of A_ν for inaccessible $\nu > \omega_0$ remains open.

1.3. In the section 2 we define an ordered set $H = R^\nu$; the properties of R^ν and of σR^ν shall yield in § 3 a requested set $A_{\nu+1}$. In § 5 we shall prove that every $A_{\nu+1}$ obtained in § 4 contains an antichain of cardinality $kA_{\nu+1}$ — a property closely connected with the author's ramification hypothesis (cf. [6], [12]).

1. The set D_ν. Let ω_ν be any initial ordinal number; we denote by D_ν the ordered set

$$\{\ldots, -\alpha, \ldots, -2, -1, 0, 1, 2, \ldots, \alpha, \ldots\} \qquad (\alpha < \omega_\nu).$$

The order type of D_ν equals $\omega_\nu^* + \omega_\nu$; i. e. $(D_\nu, \leqslant)$ is coinitial with ω_ν^* and cofinal with ω_ν. The members of D_ν might be called ω_ν-integers.

2. The sets R^ν, $R^\nu(\leqslant_n)$.

2.1. Let R^ν be the set of all *finite* sequences of members of D_ν.

2.2. Set $H=(R^\nu, \leqslant_n)$. We order R^ν by putting for members $a=(a_0, a_1 \ldots a_\alpha)$, $b=(b_0, b_1, \ldots, b_\beta)$ of R^ν that $a \leqslant_n b$ means either that a is an initial portion of b (i. e. $\alpha \leqslant \beta$ and $a_i = b_i$ for $i \leqslant \alpha$) or, that $a_i = b_i$ $(i < \varphi)$, $a_\varphi < b_\varphi$, where $\varphi = \varphi(a, b)$ is the first index at which the sequences a, b differ (*natural ordering of complexes*). One sees that

$$(2.1.) \qquad H=(R^\nu, \leqslant_n)$$

is a totally ordered set.

2.3. The cardinality of H equals $k\,\omega_\nu$, i. e. $kR^\nu = \aleph_\nu$.

As a matter of fact, $kR^\nu = \aleph_\nu + \aleph_\nu^2 + \aleph_\nu^3 + \cdots = \aleph_\nu + \aleph_\nu + \cdots = \aleph_\nu$.

2.4. Theorem (i). *The set* $H \equiv (R^\nu, \leqslant_\nu)$ *is totally ordered and dense.*

(ii) *The ordered set* H *is order-imbeddable into every of its intervals* I, *even so that* I *contains a subinterval similar with* H *(quasihomogeneity of* R^ν*).*

(iii) *Every gap* $X \mid Y$ *of* $(R^\nu, \leqslant_n)$ *is of the type* (ω_0, ω_0^*) *i. e.* X *is cofinal to* ω_0 *and* $Y = R_\nu \setminus Y$ *is coinitial to* ω_0^*. *Every interval of* H *contains gaps, i. e. the gaps of* H *are everywhere dense in* (2.1).

(IV) *Every ordinal number* α, $\alpha < \omega_{\nu+1}$ *is imbeddable into* H; $\omega_{\nu+1}$ *is the first ordinal which is not imbeddable into* H.

Proof. (1) For $a=(a_0, a_1, \ldots, a_n) \in (2.1)$, the set $S = R_\nu(\cdot, a)$ of all predecessors of a has no terminating member; in fact, let us consider $a_n \in D_\nu$; if a_n has in D_ν its immediate predecessor a_n^-, i. e. if a_n is isolated ordinal, then the set S is cofinal with the ω_ν-sequence of all sequences of the form $(a_0 a_1 \ldots a_{n-1} a_n^- \omega_\nu')$, ω_ν' running through D_ν; if $a_n^- = a_n$ (i. e. if a_n is an ordinal of the second kind) and equaling $\xi\omega$, then S is cofinal with the a_n-sequence of members $(a_0 a_1 \ldots a_{n-1} \beta)_{\beta < a_n}$. By dual considerations one proves that a has no immediate follower.

Ad (ii). Let $a=(a_0 a_1 \ldots a_m)$, $b=(b_0, b_1, \ldots, b_n) \in R^\nu$ and $a <_n b$. Then either a is an initial section of b or there is the first index $\varphi < \omega$ such that $a_\varphi < b_\varphi$ (thus $a_\varphi \neq b_\varphi$) and $a_i = b_i$ for every $i < \varphi$. In the first case, $m < n$; let c, d be two members of D_ν such that $c < d < b_{m+1}$; then the mapping $x \in R^\nu \rightarrow x' = acx$ is a requested order imbedding of (2.1) into $(R^\nu, \leqslant_n)(a, b)$, because $a <_n x' <_n b$.

In the second case, the mapping $x \in R^\nu \rightarrow ax$ furnishes such an imbedding. In both cases, the previous isomorphism carries R^ν onto some subinterval of $R^\nu(a, b)$.

Ad (iii). Let $X \mid Y$ be a section of (2.1), i. e. X is a non void initial portion of (2.1) and Y is the remainder $(2.1) \setminus X$; suppose $X \mid Y$ be a gap. Let $b_0 = \sup \xi$ where $(\xi) \in X$; then $b_0 < \omega_\nu$ because the set of all the $(\xi)\,(\xi < \omega_\nu)$ is cofinal with (2.1); let $b_1 = \sup \xi$ with $(a_0\, \xi) \in X$; if $b_1 = \omega_\nu$, then $(a_0 + 1)$ is the first member of Y.

If $b_1 < \omega_\nu$ we consider $b_2 = \sup \xi$ with $(b_0, b_1, \xi) \in X$ etc. If for some $n < \omega_0$ we have $b_n = \omega_\nu$, then $(b_0, b_1 \ldots b_{n-1} + 1)$ is the first member of Y; if $a_n < \omega_\nu$ for every $n < \omega_0$, then X is cofinal with the strictly increasing ω_0-sequence

$$(b_0),\ (b_0 b_1),\ (b_0 b_1 b_2),\ \ldots$$

and consequently has the type of ω_0.

Dual considerations show that Y is coinitial with ω_0^*. As a matter of fact, let $c_0 = \inf \xi$, where $(\xi) \in Y$. Then $-\omega_\nu < c_0$ because the set (2.1) is coinitial with the set of all sequences $(\xi)(\xi > -\omega_\nu)$. Let $c_1 = \inf \xi$ where $(c_0\, \xi) \in Y$; if $c_1 = -\omega_\nu$, then $(-1 + c_0)$ is the last member of X. If $c_1 > -\omega_\nu$, we consider $c_2 = \inf \xi$, with $(c_0\, c_1\, \xi) \in Y$, etc.

Since H has gaps, every interval of H has gaps too — consequence of the statement (ii).

Ad (iV). Every ordinal number α, $\alpha < \omega_{\nu+1}$ is imbeddable into (2.1). We prove it by induction argument. For $\alpha \leqslant \omega_\nu$, the fact is obvious: the mapping $\alpha \in I\,\omega_\nu \rightarrow (\alpha) \in R^\nu$ is such an isomorphism. Let the statement hold for any α, $\alpha < \beta$, where $\beta < \omega_{\nu+1}$; let us prove it also for $\alpha = \beta$; this being obvious for $\beta = \beta^- + 1$, let us consider the case $\beta^- = \beta$ (β is of second art). Then, there exists a regular number ω_γ such that for some ω_γ-sequence β_ξ of ordinals we have

$$\beta = \Sigma_\xi\, \beta_\xi \qquad (\xi < \omega_\gamma).$$

Now, let $b_\xi\,(\xi < \omega_\gamma)$ be any strictly increasing sequence of points of R^ν; we imbed every ordinal β_ξ into the open interval $R^\nu(b_\xi,\, b_{\xi+1})$ as some well--ordered subset B_ξ; then the union $\bigcup_\xi B_\xi\,(\xi < \omega_\gamma)$ is a well-ordered subset of (2.1) and of type β.

3. The set σ_ν**.** Let σ_ν or σ be the system of all well-ordered non void subsets $\subseteq$ (2.1), each of which is bounded; consequently, if $a \in \sigma_\nu$, the number γa — the ordinal type of a — is determined as well as the insreasing points

$$a_\xi \qquad (\xi < \gamma\, a)$$

of the set a. The system σ_ν will be ordered by the relation $=\!|$ meaning „*to be an initial segment of*“.

Of course, the set $(\sigma_\nu,\, =\!|)$ is (partially) ordered; moreover, it is ramified and even σ_ν is a tree.

3.1. Theorem. *The ordered set* $(\sigma_\nu,\, =\!|)$ *is such that each of its chains is a well-ordered set, the cardinal of which is* $\leqslant \aleph_\nu$; *on the other hand*

$$\gamma(\sigma_\nu,\, =\!|) = \omega_{\nu+1}. \tag{3.1.1}$$

Namely, if C is any chain $\subseteq (\sigma_\nu,\, =\!|)$, then $\bigcup C$ is a well-ordered subset of (2.1) and vice versa: W being any well-ordered bounded subset of (2.1), the system of non void initial intervals of W yields a chain $\subset (\sigma_\nu,\, =\!|)$. Finally, the relation (3.1.1) is an another expression for the statement 2.4. IV

3.2. The set $(R^\nu,\, \leqslant_n)$. Let

$$\overline{R^\nu} \quad \text{or} \quad \overline{H} \tag{3.2.1}$$

denote the totally ordered set obtained from (2.1) by putting a single element in every gap of (2.1). Of course, the set (3.2.1) is continuous in the sense of Dedekind: every section of (3.2.1) is given by a single element of (3.2.1). Consequently, for every bounded chain $C \subseteq$ (3.2.1) the infimum and the supremum of C relatively to (3.2.1) are well determined points of (3.2.1).

3.3. Function f.

In particular, let

$$f(a) = \sup_{x \in a} x \qquad (a \in \sigma_\nu); \tag{3.3.1}$$

3.3.1. Lemma. *The mapping* (3.3.1) *is an increasing mapping of* σ_ν *into* (3.2.1); *every* 3 *points chain of* $(\sigma_\nu, =\!|)$ *is mapped onto at least* 2 *points of* (3.2.1). *The system* $a =\!| a'$, $f(a) = f(a')$ *is equivalent with the statement that* $\sup a \in$ (3.2.1) *and* $a' \setminus a = \{\sup a\}$.

3.3.2. Lemma. *Let* $e \in \sigma_\nu$, $\beta < \omega_{\nu+1}$; *then the set* $f(R_\beta(e, \cdot)_\sigma)$ *is everywhere dense in the right interval* $(f(e), \cdot)_{\overline{H}}$.

At first, let $\beta = 0$; then $R_0(e, \cdot)$ is built up of the sets

$$e \cup \{x\},$$

x running over the set of all the points of H, each of which is $> f(e)$ or $\geqslant f(e)$, according as γe is limit or isolated ordinal; since then

$$f(e) = \sup (e \cup \{x\}) = x,$$

the statement is obvious. Let us suppose now that $0 < \beta < \omega_{\nu+1}$ and that the statement holds true for each $\xi < \beta$; to prove it for $\xi = \beta$. If $\beta - 1$ exists, the set $fR_{\beta-1}(e, \cdot)_\sigma$ is dense on $(f(e, \cdot)_H$; again, if b is an immediate successor of e in σ, then $f(R_0(e, \cdot)_\sigma)$ is dense on $(f(e), \cdot)_H$; consequently, the join

$$\cup_b f(R_0(b, \cdot)_\sigma = f(\cup_b R_0(b, \cdot)_\sigma = f(R_\beta(e; \cdot)_\sigma (b \in R_{\beta-1}(e; \cdot)_\sigma)$$

is dense on $f(e;)_H$.

If β is a limit number, let $\beta_\xi (\xi < cf\beta = \tau)$ be an increasing sequence of ordinals $\to \beta$. Let x be any point of $(f(e), \cdot)_H$ of character c_τ and $x_\eta (\eta < cf\beta)$ any increasing τ-sequence of points of H so that $f(e) < e$, $\sup\limits_{\eta<\tau} x_\eta = x$ and $\sup\limits_{\mu<\eta} x_\mu < x_\eta (\eta < \tau)$. The existence of such a chain x_η is obvious. Let $e^\circ \in R_{\beta_0}(e; \cdot)_\sigma$ so that $f(e) < f(e^\circ) < x_0$; inductively, for each $0 < \delta < \tau$ let us suppose defined the δ-chain

$$e^\mu (\mu < \delta)$$

so that $e^\mu \in R_{\beta_\mu}(e, \cdot)$, $x_\mu < f(e^\mu) < x^{\mu+1} (\mu < \delta)$. Let then e^δ be an element of $R_{\beta_\delta}(e, \cdot)_\sigma$ so that $e^\mu =\!| e^\delta$, $(\mu < \delta)$, $x_\delta < f(e^\delta) < x_{\delta+1}$; such one x_δ exists, since $x' = \bigcup\limits_{\mu<\delta} e^\mu$ is a point of σ, inasmuch it is well-ordered and bounded (it is located left to x_δ). Q. E. D.

From the last proof we deduce the following.

3.3.3. Lemma. *If* $\alpha < \omega_{\nu+1}$, *the set* $fR_\alpha \sigma_\nu$ *is everywhere dense on* $\overline{H}$; *if* α *is a limit number, then* $fR_\alpha \sigma_\nu$ *is equal to the set of all the points of* $\overline{H}$ *each of which is of character* $C_{\alpha'}$, $\alpha' = cf\alpha$.

4. Construction of the requested sets $A_{\nu+1}$.

We shall construct a requested tree $A = A_{\nu+1}$ as a union of some $k\omega_{\nu+1}$ sets $D_\xi (\xi < \omega_{\nu+1})$.

4.1. To start, let G_0 be any subset of cardinality $\aleph_\nu$ of $\overline{H} \setminus H$ so that G_0 be everywhere dense in H; consequently, every member of G_0 is of a countable character (cf. 2.4. (iii)). To every $x \in G_0$ we associate an element $\psi(x) \in R_\omega \sigma_\nu$ such that $f\psi(x) = x$ and that for $x, x' \in G_0$, $x \neq x'$ one has $\psi x \neq \psi x'$. In this way we get the set

$$D_0 := \psi G_0 \subseteq R_\omega \sigma_\nu,$$

4.2. Let suppose that $0<\beta<\omega_{\nu+1}$ and that the sets

$$D_0,\ D_1,\ \ldots,\ D_\xi,\ \ldots,\ (\xi<\beta)$$

be constructed so that putting

$$s_\beta=\bigcup_\xi D_\xi \qquad (\xi<\beta)$$

the following conditions $1_\beta - 7_\beta$ hold:

1_β $R_\xi s_\beta = D_\xi$ $(\xi<\beta)$

2_β $D_\xi \subseteq R_{\omega(1+\xi)}\sigma_\nu$ $(\xi<\beta)$

3_β $\gamma s_\beta=\beta$

4_β $kD_\xi=\aleph_\nu$ $(\xi<\beta)$

5_β If $\xi<\beta$, $e\in D_\xi$ and $\xi<\zeta<\beta$, then $fR_\xi[e]_{s_\beta}$ is an everywhere dense set on $\overline{H}(f(e),\cdot)$,

6_β If $\xi<\beta$, the set $fR_\xi s_\beta$ is everywhere dense on $\overline{H}$,

7_β For each $\xi<\beta$ and $x\in D$, fx is a $\omega(1+\xi)$ — point of $\overline{H}$; if $e, e'\in D_\xi$ and $e\neq e'$, then $fe\neq fe'$.

4.3. Let us define D_β and $s_{\beta+1}$.

4.3.1. If $\beta^-<\beta$, let us consider $D_{\beta-1}$; let l_β be a set-mapping of $D_{\beta-1}$ into $\overline{H}$ so that for each $e\in D_{\beta-1}$ the set $l_\beta e$ be a subset of cardinality $\aleph_\nu$ of ω_0-points of $fR_{\omega(1+\xi)}(e,\cdot)$ everywhere dense on it, and that if $e\neq e'$, then the sets $l_\beta e$, $l_{\beta'} e'$ are disjoint. For each $x\in l_\beta e$ let $\varphi(e, x)$ be an element of $R_{\omega\beta}[e]_{\sigma_\nu}$ such that $f\varphi(e, x)=x$ (the existence of such one ψx is obvious). Then we define

(4.3.1.1.) $$D_\beta :=\bigcup_e le \qquad (e\in D_{\beta-1}).$$

Consequently, $fD_\beta = f\bigcup_e l_\beta(e)$ = everywhere dense subset of $\overline{H}(f(e),\cdot)$.

4.3.2. Case: β is a limit ordinal. 4.3.2.1. Let β_ζ $(\zeta<\tau,\ \tau=cf\beta)$ be any increasing regular sequence of ordinals $\to\beta$; let $l_\beta e$ for

(4.3.2.1) $$e\in\bigcup_{\zeta<cf\beta}R_{\omega(1+\zeta)}s_\beta$$

be a disjointed system of sets, so that le be a subset of cardinality $\aleph_\nu$ of $\omega\tau$-points of $fR_{\omega(1+\beta)}[e]_{\sigma_\nu}$ everywhere dense on that set; in particular

(4.3.2.2) $$kle=\aleph_\nu.$$

To each ordered pair (e, x) with $x\in l_\beta(e)$ let $\psi(e, x)$ be an element of $R_{\beta(1+\zeta)}[e]_{\sigma_\nu}$ so that $f\psi(e, x)=x$. The existence of $\psi(e, x)$ is clear. As a matter of fact, x being a member in $\overline{H}$, let x_ζ $(\zeta<\tau)$ be a strictly increasing sequence of points of $(f(e),\cdot)_{\overline{H}}$ tending to x. Let e° be a point of D_{β_0} satisfying $e=|e^\circ$, $fe<fe^\circ<x_0$. Inductively, for each $0<\zeta<\tau$ and $0<\xi<\zeta$, let e^ξ be a certain point of D_{β_ξ} succeeding to every e^μ $(\mu<\xi)$ and satisfying $x_\xi<fe^\xi<x_{\xi+1}$; in virtue of conditions 4_β, 5_β the existence of e^ξ is assured. We define, $e^\zeta=\sup_{\xi<\zeta} e^\xi$ and $\psi(e, x)=\sup_{\zeta<\tau} e^\zeta$. Thence

$$f\psi(e, x)=x.$$

4.3.2.2. The set D_β is defined as consisting of points $\psi(e, x)$, x, e running respectively over $\bigcup_e l(e)$ and (4.3.2.1).

In any case, the set D_β is defined.

4.4. Putting

(4.4.1) $$s_{\beta+1} = s_\beta \cup D_\beta$$

one proves that the condititions $1_{\beta+1}$—$7_{\beta+1}$ are satisfied. The condition $1_{\beta+1}$ is satisfied since $R_\xi s_\beta = R_\xi s_{\beta+1}$ $(\xi<\beta)$ and since each $e \in D_\beta$ is preceded by a single element in every D_ξ $(\xi<\beta)$. As to $2_{\beta+1}$, its verification is immediate. As to $3_{\beta+1}$ i. e. that $kD_\beta = \aleph_\nu$, it is a consequence of the formula for D_ξ $(\xi<\beta)$, that means that $ks_\beta = \aleph_\beta$. Now, there is a one-to-$\aleph_0$-mapping of a subset[1]) of s_β onto D_β, thus $kD_\beta \leqslant ks_\beta$. $\aleph_\nu = \aleph_\nu \cdot \aleph_\nu = \aleph_\nu$; since for any $\xi<\beta$ each $e \in D_\xi$ is succeeded by $\aleph_\nu$ distinct elements of D_β, the condition $3_{\beta+1}$ is fulfilled.

$4_{\beta+1}$. The case of an isolated β being resolved by (4.4.1), (4.3.1.1), let β be limit number; x running over $l_\beta(e)$; the condition $4_{\beta+1}$ is an immediate cosequence of (4.4.1), (4.3.1.1) and of the assumed density of $l_\beta(e)$. The condition $5_{\beta+1}$ is a consequence of $4_{\beta+1}$ and of the conditions 5_ξ $(\xi<\beta)$. Finally, the condition $7_{\beta+1}$ holds true, first, because the points of D_β are constructed as some $\omega(1+\beta)$ — points of $\overline{H}$ and, secondly, because of the disjointedness of the above sets $l_\beta(e)$ (e variable). Thus the existence of D_β and $s_{\beta+1}$ is proved for every β, $\beta<\omega_{\nu+1}$ and the conditions 1_β—7_β hold true for every $\beta<\omega_{\nu+1}$.

4.5. Putting

$$A_{\nu+1} = \bigcup_\beta D_\beta \qquad (\beta<\omega_{\nu+1})$$

one sees that conditions $1\,\omega_{\nu+1}$—$6\,\omega_{\nu+1}$ hold true for writing $A_{\nu+1}$ instead of $s_{\omega_{\nu+1}}$. In particular $A_{\nu+1}$ is a tree so that

$$\gamma A_{\nu+1} = \omega_{\nu+1}$$

$$kR_\beta A_{\nu+1} = \aleph_\nu \qquad (\beta<\omega_{\nu+1})$$

$$k_c A_{\nu+1} = \aleph_\nu;$$

moreover, for any $e \in A_{\nu+1}$ the set $(e, \cdot)_A$ satisfies the same last three conditions.

4.6. Total order-extension of $A_{\nu+1}$ to become $I\,\omega_{\nu+1}$ or the partial order destroying in $I\,\omega_{\nu+1}$ to become $A_{\nu+1}$.

4.6.1. The mapping $f(e)$ $(e \in A)$ is a strongly increasing mapping of A into $\overline{H}$. Consequently, f is biunique in every chain of A. According to the previous construction, f is biunique in every set $R_\beta A$ $(\beta<\omega_{\nu+1})$. Thus, if the sets $fR_\beta A$ $(\beta<\omega_{\nu+1})$ are pairwise disjoint, f is a biunique correspondence between A and the subset fA of $\overline{H}$. In such a case, using the mapping f, we can proceed either to destroy partially the order in the chain fA and get an ordered set similar to A, or to transfer the total order of fA onto the set A enlarging so the given partial order of A. Namely, if e, e' are any two incomparable points of A, we can declare incomparable also the corresponding points fe, fe' in fA; fA becomes partially ordered and similar to A. And vice versa, if x, x' are any two points of fA, we can introduce an order $<'$ into A by the procedure that $x<x'$ in fA be equivalent to $f^{-1}x <' f^{-1}x'$ in A.

[1]) Viz. of $D_{\beta-1}$ and $\bigcup_{\varsigma<cf\,\beta} D_{\beta_\varsigma}$ respectively, according as β in isolated or limit number.

4.6.2. *Partial desordonning of $I(\omega_{\nu+1})$ to get a set $A_{\nu+1}$.* Any set $A_{\nu+1}$ has $\aleph_{\nu+1}$ as its cardinality. We can do an extension of order of A to yield a *total order of type* $\omega_{\nu+1}$ of $A_{\nu+1}$. Namely, as $kR_\beta A=\aleph_\nu$, let

$$a_\xi^\beta\,(\xi<\omega_\nu)$$

be an ω_ν-enumeration and at the same time an ordering of the set $R_\beta A$, so that in the new ordering a_ξ^β precedes $a_{\xi'}^\beta$ if and only if $\xi<\xi'<\omega_\nu$. Defining $a_\xi^\beta \leqslant a_{\xi'}^{\beta'}$ if and only if either $\beta\leqslant\beta'$ or $\beta=\beta$, $\xi\leqslant\xi'$ one gets the required total extension of $(A, \leqslant)$.

Putting

$$g\left(a_\xi^\beta\right)=\omega_\nu\beta+\xi\,(\beta<\omega_{\nu+1},\ \xi<\omega_\nu)$$

g is a biunique isomorphic mapping of $A_{\nu+1}$ onto $I(\omega_{\nu+1})$. This isomorphism enables us to destroy partially the total order in $I(\omega_{\nu+1})$ to get in it, as a step of previous ordination of I the partial order of A.[1]

4.6.3. It is not easy to have a simple picture how such an desordonning of $I(\omega_{\nu+1})$ takes place. However, we can realize it in the following manner: let $h(\beta)\,(\beta<\omega_{\nu+1})$ be any uniform mapping of $I(\omega_{\nu+1})$ into $\overline{H}$ so that for each β the mapping h be biunique in $[\omega_\nu\beta,\ \omega_\nu(\beta+1))$ and that the corresponding set

$$h\,[\omega_\nu\beta,\ \omega_\nu(\beta+1)) \tag{4.6.3.1}$$

be everywhere dense in $\overline{H}$ and be composed of very $\omega(1+\beta)$ — points of $\overline{H}$; then the set (4.6.3.1) can be chosen to serve as the set fD_β in the construction in § 4.3: it is sufficient to consider any partition P_β of (4.6.3.1) into $\aleph_\nu$ pairwise disjoint sets, each of which is everywhere dense, establish a biunique correspondence t_β between E_β and P_β and for any $e\in E_\beta$ put $l(e)=t_\beta(e)$. According as β is isolated or limit number, the set E_β means $D_{\beta-1}$ or $\bigcup_{\zeta<\tau} D_{\beta_\zeta}$ in previous notations. On the other hand, the existence of such mappings h is easy to establish. Namely, let us consider a bounded ω_ν-sequence a in H; then for each $\zeta\leqslant\nu$ the element $\sup_{\xi<\omega_\zeta} a_\xi$ is a well--determined ω_ζ-point of $\overline{H}$; thus, there are ω_ζ-points in $\overline{H}$ for each $\zeta\leqslant\nu$. In virtue of the quasihomogeneity of H that means that in each interval of $\overline{H}$ there are ω_ζ-points too; thus, for each $\beta<\omega_{\nu+1}$ the $\omega(1+\beta)$ — points are everywhere dense. It is then sufficient to consider a set S_β of power $\aleph_\nu$ of $\omega_\zeta(1+\beta)$ — points everywhere dense, decompose it into a ω_ν-system of disjoint sets $S_\xi^{\beta'}\ \beta'<\omega_\nu)$, each of which is everywhere dense and to consider the sets

$$S_\beta^{\beta'}\,(\beta<\omega_\nu,\ \beta'<\omega_\nu).$$

They are to be used as sets fD_ξ in §§ 4.1.—4.3.

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[1] The precise definition of that idea is the following one [7]: let $(E_1, \leqslant_1)(E_2, \leqslant_2)$ be two ordered sets (in general, they are only partially ordered); of course, one can have $E_1=E_2$; we say that the order of $(E_1;\leqslant_1)$ is *at least equal to the order* $(E_2;\leqslant_2)$, symbolically $t(E_1;\leqslant_1)\leqslant t(E_2,\leqslant_2)$ if there is a one-to one mapping f of E_1 into E_2 so that *every chain* C *of* $(E_1;\leqslant_1)$ is mapped onto a *similar chain of* $(E_2;\leqslant_2)$, no *matter what happens with subsets of* E_1 *that are no chains*. We say that the *ramification* (*or disorder*) *in* $(E_1,\leqslant_1)$ is at least equal to the ramification (or disorder) of $(E_2;\leqslant_2)$, symbolically $r(E_1,\leqslant_1)\leqslant r(E_2,\leqslant_2)$ if there is a one-to-one mapping of E_1 into E_2 so that if $x, y\in E_1$ and $x\leqslant_1 y$, $x>_1 y$, $x\|\leqslant_1 y$ then in $(E_2;\leqslant_2)$ respectively $f(x)\leqslant_2 f(y)$, $f(x)>_2 f(y)$, $f(x)\|\leqslant_2 f(y)$ i. e. if $(E_1;\leqslant_1)$ is similar with a subset of $(E_2;\leqslant_2)$.

5. Normality of the preceding set $A_{\nu+1}$.

Theorem. *The set $A_{\nu+1}$ of the foregoing construction contains a set of $kA_{\nu+1}=\aleph_{\nu+1}$ pairwise incomparable points.*

To see it (cf. Kurepa [10]) let $r\in H$ and $a\in A$; let us define $\psi(r, a)$ so that:

(5.1) $$\psi(r, a)=-1 \qquad \text{if } a \text{ non } \in a$$

$$\psi(r, a)=\beta \qquad \text{if } r\in a \text{ and}$$

just $a_\beta=r$ (let us remind that a is a well-ordered set $\subseteq H$). For any $T\subseteq\sigma H$ let

(5.2) $$\psi(r, T)=\sup_{a\in T}\psi(r, a).$$

There is a point $r_0\in H$ so that

(5.3) $$\psi(r_0, A)=\omega_{\nu+1}.$$

In opposite case, one should have $\psi(r; A)<\omega_{\nu+1}(r\in H)$, thus $\delta<\omega_{\nu+1}$, with $\delta=\sup_{r\in H}\psi(r; A)$, because $kH=\aleph_\nu$. Now, since $\gamma A=\omega_{\nu+1}$ and $\delta<\omega_{\nu+1}$, there is an element $a\in R_{\delta+1}A$; the point $a_{\delta+2}$ of H should satisfy $\psi(a_{\delta+2} a)>$ $>\delta+2>\delta$, what is a nonsense. Thus the existence of an r_0 satisfying (5.3) is assured.

Now, let us construct a $\omega_{\nu+1}$-sequence

(5.4) $$a^\circ, a', \ldots, a^\xi, \ldots, \quad (\xi<\omega_{\nu+1})$$

of points of A, so that $r_0\in a^\xi(\xi<\omega_{\nu+1})$ and that for each ξ one has

(5.5) $$\psi(r_0, a^\xi)>\sup_{\zeta<\xi}\psi(r_0, a^\zeta).$$

Because of (5.3) the existence of (5.4) is inductively provable. Since, each a^ξ is a well-ordered subset of H containing r_0 and since (5.5) holds, the elements (5.4) are pairwise incomparable: no one is an initial interval of another one Q. E. D.

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