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ON THE CATEGORY NUMBER OF TOPOLOGICAL SPACES

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Introduction. René Baire introduced the notion of category of a space S and defined that S is of the first or of the second category, according as the space S is or is not a union of $\leqslant \aleph_0$ nowhere dense sets. In this connexion we consider the *minimal number* of nowhere dense sets exhausting the derivative DS of the space S. Also we were lead to consider the *minimal number* of nowhere dense sets the union of which should be everywhere dense in DS.

We shall examine how these notions are connected with some other properties of space. We consider also the class of all corresponding category numbers for some classes of spaces.

1.1. **Definition of the category number** ctS **of** S. Let be: S a topological space, IS the set of all isolated points and DS the derivative of S (i. e. $DS := S \setminus IS$). The minimal number n such that there are n nowhere dense sets the union of which coincides with DS is called the category (*number*) *of* S and is denoted by ctS; in other words

$$ctS = \inf_Y kY, \quad \bigcup_{X \in Y} X = DS, \quad X \in Y \Rightarrow S = \overline{CX}.$$

In particular we put $ct\,\emptyset = 0$.

1.2. **Definition of the subcategoricity** $sctS$ **of** S. The infimum of cardinalities of set systems Y of nowhere dense sets, the union of which yields an everywhere dense set on DS is called *the subcategory number of* S and is denoted by $cstS$.

For example, if S is the real line or any complete dense metrical space then $sctS = \aleph_0$ and $ctS > \aleph_0$; if $2^{\aleph_0} = \aleph_1$, then $ctR = \aleph_1$.

If sS denotes the *separability* of S (i. e. sS is the infimum of cardinalities of everywhere dense sets) then obviously

$$sctS \leqslant sS.$$

We shall now exhibit a class of totally ordered spaces showing that sct S, ctS might be $\aleph_0$, while kS, sS are arbitrarily high.

2. **Transition** $(O, \leqslant) \to (\sigma O, =\!|)$ **for any ordered set** $(O, \leqslant)$.

2.1. We define σO as the set of all well ordered subsets X of $(O, \leqslant)$ such that $Y \leqslant y$ for some $y \in O$. For $a, b \in \sigma O$ we denote $a =\!| b$ the fact that

a is an initial section of b. If L is any ordered chain we extend the partial order of $(\sigma L, =\!|)$ into a total order $\leqslant_n$ defining $a \leqslant_n b$ to mean $a =\!| b$ or $a_e < b_e$ in $(L, \leqslant)$ for any $a, b \in (\sigma L, =\!|)$; here $a = \{a_0, a_1, \ldots\}$, $b = \{b_0, b_1, \ldots\}$, $a_e < b_e$, $e := e(a, b)$ denoting the first ordinal number ξ such that $a_\xi \neq b_\xi$.

2.2. We proved (Đ. Kurepa [2] that $(\sigma L, \leqslant_n)$ is a linearly ordered set. Consequently, for any totally ordered set $(L, \leqslant)$we have also the ordered space $(\sigma L, \leqslant_n)$ defined by the interval topology on $(\sigma L, \leqslant_n)$.

2.3. Let us consider the particular example of the ordered set $(Q, \leqslant)$ of rational numbers and of its order type η_0. We use a similar terminology for any ordered set and for its order type.

2.4. Lemma. *Let $x \in \eta_0$; then the totally ordered subset*

$$(1) \qquad (\sigma\eta_0(\cdot, x), \leqslant_n)$$

is nowhere dense in the chain

$$(2) \qquad (\sigma\eta_0; \leqslant_n), \text{ i.e.}$$

every non void interval I of the chain (2) *contains an interval I_0 disjoint from the chain* (1).

Proof. First (cf. Đ. Kurepa [3] p. 91 Lemma 4.2 and [4] L. 4.2. p. 41) there exists some $c \in \sigma\eta_0$ such that

$$(\sigma\eta_0[c, \cdot) =\!|) \subseteq I.$$

Since for every $c -\!| c' \in \sigma\eta_0$ we have also $\sigma\eta_0[c', \cdot) \subset I$ we might suppose that the set c contains at least one point c_ν such that $x < c_\nu$ and $c = \{c_\xi\}_{\xi < \gamma c}$; γc denotes the order type of the well ordered set c.

Now, let $a \in \sigma(\eta_0(\cdot, x))$. Then $e(a, c) < \nu$. According as $a \leqslant_n c$ or $a \geqslant_n c$ we have $a <_n \sigma\eta_0[c, \cdot)$ or $a >_n \sigma\eta_0[c, \cdot)$ and never $a \in \sigma(\eta_0[c, \cdot) -\!|) = A$. Since A is a non empty section of the chain $(\sigma\eta_0, \leqslant_n)$, every interval it contains is disjoint from $(\sigma\eta_0(\cdot, x), \leqslant_n)$ what was to be proved.

2.5. Theorem. *Let $(L, \leqslant)$ be any dense ordered chain; then the category $ct(\sigma(L, \leqslant), \leqslant_n)$ and the cofinality number $cf(L, \leqslant)$ satisfy the relation*

$$ct(\sigma(L, \leqslant) \leqslant_n) \leqslant cf(L, \leqslant).$$

As a matter of fact, let W be any well ordered subset of $(L, \leqslant)$ such that $kW = cfL$ and that $(W, \leqslant)$ should be cofinal to $(L, \leqslant)$. Then $(\sigma L, \leqslant_n)$ is the union of kW nowhere dense sets $\sigma L(\cdot, x)$ $(x \in W)$.

2.6. Theorem. (i) *The category number of the chain* (1) $\sigma(\eta_0 \leqslant_n)$ *is* $\aleph_0$. *If L is any ordered chain cofinal with ω_0, then the category number of the ordered space $(\sigma\omega_0, \leqslant_n)$ equals $\aleph_0$.*

(ii). *The chain* (1) *is not a union of $\aleph_0$ special nowhere dense sets, each being an antichain in the tree $(\sigma\eta_0, =\!|)$.*

The first part (i) is implied by the theorem 2.5. As to the second part (ii) we refer to the theorem 4.1 p. 37 of the paper Đ. Kurepa [4].

2.7. Corollary. *There are spaces of the categoricity $\aleph_0$ of any high cardinality $\geqslant a$ and of any separability degree $s \geqslant \aleph_0$.*

As a matter of fact, it is sufficient to consider any ordered chain L such that $cfL = \aleph_0$, $kL \geqslant a$; then $cf(\sigma L, \leqslant) = \aleph_0$, $k\sigma L \geqslant a$. Of course, we might suppose also that sep $L \geqslant s$; this with sep $\sigma L \geqslant$ sep L implies all requested conditions.

3. By an analogous argument one proves the following.

3.1. Theorem. *If $(L, \leqslant)$ is any dense ordered chain cofinal to a regular initial number ω_δ, then the categoricity of the natural order*

(1) $$(\sigma(L, \leqslant), \leqslant_n) \text{ is } \leqslant \aleph_\delta:$$

(2) $$ct(\sigma(L, \leqslant), \leqslant_n) \leqslant \aleph_\delta,$$

in particular if S is any subset of L cofinal to S and to ω_δ, then the sets

(3) $$(\sigma(S(\cdot, x)) \leqslant_n) \ (x \in S)$$

are nowhere dense in (3) and exhaust the chain (1).

3.2, Corollary. *For every regular ordinal number ω_δ one has*

$$ct(\sigma\eta_\delta, \leqslant_n) \leqslant \aleph_\delta.$$

4. Some propositions concerning the category numbers.

4.1. Theorem. *For a given infinite number $ctS = n$ and any infinite cardinal number $m > n$ there exists a space X satisfying $ctX = n$, $kX = m$.*

As a matter of fact, it suffices to consider a family F of cardinality m of replicas of S and the union U of all the members of F topologysed in such a way that every member X of F be the subspace in U identical with the given space X. Then $ctU = ctS$, $kU = mkS = m \cdot n = m$.

4.2. Theorem. *For every infinite number b there exists a metrical space M_b satisfying $ctM = b$.*

As a matter of fact, let $R(b)$ be the set of all the ω-sequences of ordinals $< \omega(b)$, each of which is almost the constant sequence 0, 0 . . ; we metrisize $R(b)$ by the function $\rho(a, b) = \dfrac{1}{1 + \varphi(a, b)}$ for every pair $a = (a_0, a_1, \ldots)$, $b = (b_0, b_1, \ldots)$ of distinct members of $R(b)$; $\varphi(a, b)$ is the first index n at which $a_n \neq b_n$.

Then the space $(R(b), \rho)$ is metrical, dense in itself and satisfies $sep R(b) = b = kR(b) = ctR(b)$.

4.2.1. Corollary. *M running through the class of metrical spaces the class $\{ctM \ldots\}_M$ is not a set, because it contains every infinite cardinal number b.*

4.3. Problem. Let S be any topological space; if $ctS > \aleph_0$ and if n is any cardinal number satisfying $\aleph_0 \leqslant n < ctS$, is there a subspace S' of S satisfying $ctS' = n$?. In other words, do the numbers ctX, $(X \subseteq S)$ fill the cardinal interval $[sep\, S,\ ct\, S]$?

4.3.1. For metrical spaces, the general continuum hypothesis implies a positive answer.

4.3.2. The question arises to establish the result 4.3.1. without assuming the continuum hypothesis.

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