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ON FIRST INTEGRALS OF THE EQUATIONS OF MOTION OF THE FORM $\frac{dx^i}{dt}=a^{ij}\frac{\partial \Phi}{\partial x^j}$

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The differential equations of geodesics in the Riemannian space $\overline{V}_n$ with the action line-element

$$ds^2=g_{ij}\,dx^i\,dx^j, \qquad (i, j=1, 2, \ldots\ldots\ldots, n) \tag{1}$$

where g_{ij} is the metric tensor of the space $\overline{V}_n$, admit a complete system of n-first linear integrals [1], in the form of

$$\frac{dx^i}{ds}=g^{ij}\frac{\partial \Phi}{\partial x^j}, \qquad (i, j=1, 2, \ldots\ldots\ldots, n) \tag{2}$$

provided the Riemannian space $\overline{V}_n$ admits the solutions

$$\Phi=\Phi\,(x^1, x^2, \ldots\ldots, x^n;\ C_1, C_2, \ldots\ldots, C_n), \tag{3}$$

of the partial differential equation

$$g^{ij}\frac{\partial \Phi}{\partial x^i}\frac{\partial \Phi}{\partial x^j}=1. \tag{4}$$

Application of this theorem to the conservative systems is obvious. If a_{ij} is the fundamentaltens or corresponding to the kinematic line-element for a material system with the kinetic energy $T=\frac{1}{2}a_{ij}\dot{x}^i\dot{x}^j$, and if E is the total and V the potential energy of this system, so that $g_{ij}=2(E-V)\,a_{ij}$, [2] represents the action line-element for this system, it follows directly from (2) that

$$\frac{dx^i}{dt}=a^{ij}\frac{\partial \Phi}{\partial x^j}. \tag{5}$$

The question now is whether the differential equations of motion of a scleronomic non-conservative system in the Riemannian space V_n admit first integrals which are linear in terms of the generalized velocities of the form (5). In order to answer this question, we shall consider a scleronomic holonomic non-conservative system which is acted upon by some given forces which are functions of positions only. Let x^i be generalized coordinates of such a system, for which we assume that it has n-degrees of freedom. The differential equations of motion of the scleronomic non-concervative system under consideration can be taken to represents the motion of a particle in the Riemannian space V_n and are as follows:

$$\frac{d^2 x^i}{dt^2}+\left\{{i \atop j\ k}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}=Q^i(x), \tag{6}$$

where $\left\{{i \atop j\ k}\right\}$ are Christoffel's symbols of the second kind formulated with respect to the metric tensor a_{ij} of the space V_n which is defined from the relation

$$ds^2=2\,T\,dt^2=a_{ij}\,dx^i\,dx^j \qquad (i, j=1, 2, \ldots, n). \tag{7}$$

In order to see whether the differential equations of the scleronomic non-conservative system (6) in the Riemannian space V_n admit first integrals of the form (5), we shall assume that the generalized velocities $T^i \equiv \frac{dx^i}{dt}$, in (6) can be given as

$$\frac{dx^i}{dt}=T^i=a^{ij}\frac{\partial\Phi}{\partial x^j}. \tag{8}$$

The substitution of (8) in (6) will result in

$$\frac{d}{dt}T^i+\left\{{i \atop j\ k}\right\}T^j\,T^k=Q^i(x). \tag{9}$$

When the differentiation indicated in (9) is carried out, and by taking into consideration (8), it follows that

$$a^{il}\,a^{mn}\frac{\partial^2\Phi}{\partial x^l\,\partial x^n}\frac{\partial\Phi}{\partial x^m}+\frac{\partial\Phi}{\partial x^m}\frac{\partial\Phi}{\partial x^n}\,a^{pm}\left(\frac{\partial a^{in}}{\partial x^p}+a^{kn}\left\{{i \atop p\ k}\right\}\right)=Q^i(x). \tag{10}$$

If we now start from the fact that the covariant derivative of the metric tensor a^{in} in the Riemannian space V_n is equal to zero, i. e.,

$$\nabla_p\,a^{in}=\frac{\partial a^{in}}{\partial x^p}+a^{kn}\left\{{i \atop j\ k}\right\}+a^{ik}\left\{{n \atop p\ k}\right\}=0, \tag{11}$$

(where the symbol ∇ denotes a covariant derivative), then the expression in brackets in (10) can be replaced with

$$-a^{ik}\left\{{n \atop p\ k}\right\}$$

which follows from (11). When this substitution is made, the equations (10) become

$$a^{il}\,a^{pm}\frac{\partial^2\Phi}{\partial x^l\,\partial x^p}\frac{\partial\Phi}{\partial x^m}-\left\{{n \atop p\ l}\right\}\frac{\partial\Phi}{\partial x^m}\frac{\partial\Phi}{\partial x^n}\,a^{pm}\,a^{il}=Q^i(x) \tag{12}$$

or

$$a^{il}\,a^{pm}\frac{\partial\Phi}{\partial x^m}\left(\frac{\partial}{\partial x^l}\frac{\partial\Phi}{\partial x^p}-\frac{\partial\Phi}{\partial x^n}\left\{{n \atop p\ l}\right\}\right)=Q^i(x). \tag{13}$$

In (13), the expression in brackets represents the covariant derivative of the vector $\frac{\partial \Phi}{\partial x^p}$, thus the equations (13) can be written as follows

$$a^{pm} \frac{\partial \Phi}{\partial x^m} \nabla_k \frac{\partial \Phi}{\partial x^p} = Q_k (x), \tag{14}$$

which leads to

$$\nabla_k \left(a^{pm} \frac{\partial \Phi}{\partial x^p} \frac{\partial \Phi}{\partial x^m} \right) = 2\, Q_k (x). \tag{15}$$

Since the expression in brackets in (15) is as scalar quantity, the covariant derivative is equal to the partial derivative, and the equations (15) can now be written as

$$\frac{\partial}{\partial x^k} \left(a^{pm} \frac{\partial \Phi}{\partial x^p} \frac{\partial \Phi}{\partial x^m} \right) = 2\, Q_k (x), \tag{16}$$

which, in view (7), leads us to the following expression

$$\frac{\partial}{\partial x^k} T = Q_k (x). \tag{17}$$

Since generalized forces, however, are not conservative, then

$$Q_k (x) \neq \frac{\partial U}{\partial x^k}$$

where U — is the force function, and thus, the relation (17) is impossible. Hence, we have the following theorem:

The scleronomic non-conservative system, the differential equations of motion of which are (6), do not admit first integrals of the form (5) in the Riemannian space V_n.

REFERENCES

[1] Lazar Rusov, *On some first integrals of equations of motion*, Publ. Inst. Math. Beograd 5 (19), 1965.

[2] Tatomir Anđelić, *Tenzorski račun (Tensor Calculus)*, II izd. Naučna knjiga, Beograd 1967.