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TWO PROPERTIES OF BORN'S RELATIVISTICALLY RIGID BODY

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Born's definition of a relativistically rigid body requires the constancy of an interval between world lines of any two adjacent particles of the body during the motion, the interval being orthogonal to the world lines. In this paper we shall show that this definition gives two following consequences:

1. During the motion, an area of a triangle, with vertices in an event on a world line of one particle of the body and in events in which the null hypersurface of the first event is intersected by a world line of an other, adjacent particle, is constant.

2. During the motion, an interval between an event on a world line of one particle of the body and an event on a world line of an other, adjacent particle, dividing in a given proportion the segment determined by the events in which the world line of the second particle intersects a null hypersurface of the event of the world line of the first particle, is constant.

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Let us consider[1)] a system of particles C_{ξ^i} $(i=1, 2, 3)$, where the ξ^i are parameters characterizing individual particles of the system. Let

$$x^\alpha = x^\alpha(\xi^i, \theta) \tag{1}$$

be equations determining motions of particles C_{ξ^i} of the system with respect to system of coordinates x^α $(\alpha=1, 2, 3, 4)$ of the space-time, θ being any timelike parameter. Throughout, Greek indices will take the values 1, 2, 3, 4, and the Lation ones 1, 2, 3. Let $g_{\alpha\beta}$ be a metric tensor in the x^α system of coordinates, such that the signature of the form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \tag{2}$$

is 2, and that a spacelike interval defined by the form is positive.

The four-velocity vector is

$$U^\alpha = \frac{\partial x^\alpha}{\partial \theta}, \tag{3}$$

and the unit four-velocity vector is

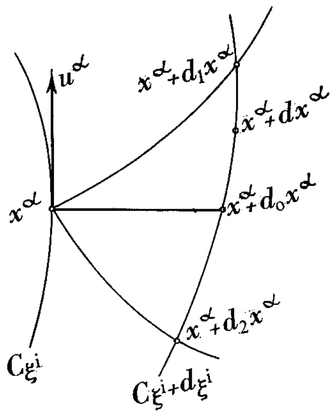
$$u^\alpha = (-U_\lambda U^\lambda)^{-\frac{1}{2}} U^\alpha \qquad (U_\lambda = g_{\lambda\mu} U^\mu). \tag{4}$$

[1)] We follow the notation Salzman and Taub used in their paper, Phys. Rev., **95**, p. 1659 (1954).

For the later we have

(5) $$u_\alpha u^\alpha = -1.$$

Let C_{ξ^i} and $C_{\xi^i+d\xi^i}$ be world lines of two adjacent particles (see the figure), and x^α an event on the world line C_{ξ^i}. Let $x^\alpha+d_1x^\alpha$ and $x^\alpha+d_2x^\alpha$ be events in which $C_{\xi^i+d\xi^i}$ intersects a null hypersurface of the event x^α, and $x^\alpha+d_0x^\alpha$ an event on the $C_{\xi^i+d\xi^i}$ such that d_0x^α is orthogonal to the C_{ξ^i}, i.e. to u^α in x^α. Born's definition equires



6) $$\frac{\partial}{\partial\theta} d_0 l^2 = 0,$$

where

(7) $$d_0 l^2 = g_{\alpha\beta} d_0 x^\alpha d_0 x^\beta.$$

Salzman and Taub showed[1)] that

(8) $$d_0 l^2 = (g_{\alpha\beta} + u_\alpha u_\beta)\, x^\alpha_{,i}\, x^\beta_{,j}\, d\xi^i d\xi^j. \quad \left(x^\alpha_{,i} \equiv \frac{\partial x^\alpha}{\partial \xi^i}\right)$$

Let us calculate an area of a triangle with vertices in x^α, $x^\alpha+d_1x^\alpha$, and $x^\alpha+d_2x^\alpha$.

An area P of a triangle, determined by infinitesimal vectors a^α and b^α in a point x^α of a Riemannian space, is given by

(9) $$(2P)^2 = \mathrm{mod}\left(\begin{vmatrix} g_{\alpha\lambda} g_{\alpha\mu} \\ g_{\beta\lambda} g_{\beta\mu} \end{vmatrix} C^{\alpha\beta} C^{\lambda\mu}\right),$$

mod signifying modulus, where

(10) $$C^{\alpha\beta} = a^{[\alpha} b^{\beta]} \equiv \frac{1}{2}(a^\alpha b^\beta - a^\beta b^\alpha)$$

is a simple bivector — an alternating product of a^α and b^α.

In order to make easier the calculation of the area of our triangle, determined by the vectors d_1x^α and d_2x^β (see the figure), we shall derive another form of the formula (9). Replacing (10) in (9), it is easy to obtain

(11) $$(2P)^2 = |\, a_\alpha a^\alpha b_\beta b^\beta - a_\alpha b^\alpha a_\beta b^\beta \,|.$$

In the special case of a Riemannian space with an indefinite metric, when at least one of the vectors a^α and b^α is isotropic (this is just the case with both vectors d_1x^α and d_2x^α), (11) reduces to

(12) $$(2P)^2 = (a_\alpha b^\alpha)^2,$$

i.e. to

(13) $$2P = |\, a_\alpha b^\alpha \,|.$$

The area P of our triangle is, consequently,

(14) $$2P = |\, g_{\alpha\beta} d_1 x^\alpha d_2 x^\beta \,|.$$

From (1) we have

(15) $$dx^\alpha = x^\alpha_{,i} d\xi^i + U^\alpha d\theta.$$

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2) Op. cit.

$d_1 x^\alpha$ and $d_2 x^\alpha$ are solutions of the equation

$$g_{\alpha\beta} dx^\alpha dx^\beta = 0, \tag{16}$$

or

$$d_1 x^\alpha = x^\alpha_{,i} d\xi^i + U^\alpha d_1\theta, \tag{17}$$

$$d_2 x^\alpha = x^\alpha_{,i} d\xi^i + U^\alpha d_2\theta, \tag{18}$$

where $d_1\theta$ and $d_2\theta$ are solutions of (16) when dx^α is replaced in it by (15), i.e. solutions of

$$U_\alpha U^\alpha (d\theta)^2 + 2U_\alpha x^\alpha_{,i} d\xi^i \cdot d\theta + g_{\alpha\beta} x^\alpha_{,i} x^\beta_{,j} d\xi^i d\xi^j = 0. \tag{19}$$

Substituting in (14) $d_1 x^\alpha$ and $d_2 x^\alpha$ from (17) and (18) we get

$$2P = | g_{\alpha\beta} x^\alpha_{,i} x^\beta_{,j} d\xi^i d\xi^j + U_\alpha x^\alpha_{,i} d\xi^i (d_1\theta + d_2\theta) + U_\alpha U^\alpha d_1\theta\, d_2\theta |. \tag{20}$$

From (19) we have

$$d_1\theta + d_2\theta = 2\frac{U_\alpha x^\alpha_{,i} d\xi^i}{-U_\beta U^\beta}, \tag{21}$$

and

$$d_1\theta\, d_2\theta = \frac{g_{\alpha\beta} x^\alpha_{,j} x^\beta_{,i} d\xi^i d\xi^j}{U_\gamma U^\gamma}, \tag{22}$$

and (20), because of (4), turns to

$$P = | (g_{\alpha\beta} + u_\alpha u_\beta) x^\alpha_{,i} x^\beta_{,j} d\xi^i d\xi^j |. \tag{23}$$

Referring to (8) and to the fact that $d_0 l^2 > 0$, because $d_0 x^\alpha$, as a vector orthogonal to the timelike vector u^α, is a spacelike vector, we get at last

$$P = d_0 l^2, \tag{24}$$

which proves that for Born's relativistically rigid body the area of the considered triangle is constant during the motion.

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Now, let us observe on the world line $C_{\xi^i + d\xi^i}$ an event $x^\alpha + dx^\alpha$ (see the figure) dividing the segment determined by the events $x^\alpha + d_1 x^\alpha$ and $x^\alpha + d_2 x^\alpha$ in a given constant proportion k. We shall show that for Born's relativistically rigid body the interval between x^α and $x^\alpha + dx^\alpha$, i.e. the magnitude of the vector dx^α, is constant during the motion.

From the required proportion it follows that

$$dx^\alpha = \frac{d_1 x^\alpha + k d_2 x^\alpha}{1+k}, \tag{25}$$

or, referring to (17) and (18),

$$dx^\alpha = x^\alpha_{,i} d\xi^i + U^\alpha \frac{d_1\theta + k d_2\theta}{1+k}. \tag{26}$$

From (19) we get for $d_1\theta$ and $d_2\theta$

$$d_{1,2}\theta = \frac{-U_\alpha x^\alpha_{,i} d\xi^i \pm \sqrt{U_\alpha U_\beta x^\alpha_{,i} x^\beta_{,j} d\xi^i d\xi^j - U_\alpha U^\alpha g_{\beta\gamma} x^\beta_{,i} x^\gamma_{,j} d\xi^i d\xi^j}}{U_\lambda U^\lambda}, \tag{27}$$

(28) $$d_{1,2}\theta = \frac{-U_\alpha x^\alpha_{,i}\, d\xi^i \pm \sqrt{D}}{U_\lambda U^\lambda},$$

where the radicand D can be written, by suitable changes of dummies, in the form

(29) $$D = (U_\alpha U_\beta - U_\gamma U^\gamma g_{\alpha\beta})\, x^\alpha_{,i}\, x^\beta_{,j}\, d\xi^i\, d\xi^j.$$

Substituting in (26) $d_{1,2}\theta$ from (28), we get

(30) $$dx^\alpha = x^\alpha_{,i}\, d\xi^i + U^\alpha \frac{U_\lambda x^\lambda_{,i}\, d\xi^i}{-U_\mu U^\mu} - \frac{1-k}{1+k} U^\alpha \frac{\sqrt{D}}{-U_\mu U^\mu},$$

or, referring to (4),

(31) $$dx^\alpha = x^\alpha_{,i}\, d\xi^i + u^\alpha u_\lambda x^\lambda_{,i}\, d\xi^i - \frac{1-k}{1+k} u^\alpha \sqrt{\Delta},$$

where

(32) $$\Delta \equiv (g_{\alpha\beta} + u_\alpha u_\beta)\, x^\alpha_{,i}\, x^\beta_{,j}\, d\xi^i d\xi^j = d_0 l^2.$$

Now, the interval between x^α and $x^\alpha + dx^\alpha$ is

(33) $$dl^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta.$$

Putting here dx^α from (31) and taking account of (5) we get

(34) $$dl^2 = (g_{\alpha\beta} + u_\alpha u_\beta)\, x^\alpha_{,i}\, x^\beta_{,j}\, d\xi^i d\xi^j - \left(\frac{1-k}{1+k}\right)^2 \Delta.$$

With regard to (8) and (32) we have, at last,

(35) $$dl^2 = \frac{4k}{(1+k)^2}\, d_0 l^2,$$

wherefrom it is obvious that for Born's body dl^2 is constant during the motion, which proves the statement.

For $k=1$, i.e. for the event at the middle of the segment between $x^\alpha + d_1 x^\alpha$ and $x^\alpha + d_2 x^\alpha$, from (35) it follows

(36) $$dl^2 = d_0 l^2,$$

which was directly demonstrated in an earlier paper[1]. For $k<0$ it follows form (35) that dx^α is a timelike vector, which is expected because then the event $x^\alpha + dx^\alpha$ is out of the mentioned segment. For $k=0$ we have $dl^2=0$, and for $k \to \pm\infty$ we get $dl^2 \to 0$. These results are expected too, because dx^α reduces then to $d_1 x^\alpha$, i.e. it tends to $d_2 x^\alpha$.

[1] Publ. Inst. Math. 1 (15) (1961), p. 25 (1962).