PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 8 (22), 1968, p. 100—101.

CLUSTERS AND ULTRAFILTERS

Brian Warrack and Somashekhar Naimpally

(Received March 31, 1967)

Leader ([1], [2]) has shown the importance of *clusters* in the theory of proximity spaces. Several proofs of fundamental theorems cocerning clusters involve constructions of ultrafilters, and even the definition of a cluster reminds one of the characteristic properties of an ultrafilter with the difference that *nearness* replaces *nonvoid intersection*. In this note we show the interdependence of clusters and ultrafilters which throws light on several proofs besides simplifying them considerably by avoiding the direct use of the separation axiom (4) (Leader [1] page 206). We use Leader's notation. Let X denote a nonempty set.

The following lemma on ultrafilters will be used in proving our fundamental lemma (3).

1. Lemma: Let $\boldsymbol{P}$ be a property, not satisfied by Φ, such that for subsets A, B of X, $(A \cup B)$ has $\boldsymbol{P}$ iff A has $\boldsymbol{P}$ or B has $\boldsymbol{P}$. Let A_0 be an arbitrary subset of X with the property $\boldsymbol{P}$. Then there exists an ultrafilter $\mathcal{U}$ in X such that (a) $A_0 \in \mathcal{U}$ (b) $A_i \in \mathcal{U}$ for $i = 1, \ldots, n$ implies $\bigcap_{i=1}^{n} A_i$ has $\boldsymbol{P}$.

Proof: By Tukey's Lemma there exists a maximal collection $\mathcal{U}$ satisfying (a) and (b). $\mathcal{U}$ is clearly a filter. We show that $\mathcal{U}$ is an ultrafilter. If not, then there exists an $E \subset X$ such that neither E nor $X—E$ is in $\mathcal{U}$. Hence there are A_1, A_2 in $\mathcal{U}$ such that neither $A_1 \cap E$ nor $A_2 \cap (X—E)$ has $\boldsymbol{P}$. If $A = A_1 \cap A_2$, then A has $\boldsymbol{P}$ but neither $A \cap E$ nor $A \cap (X—E)$ has $\boldsymbol{P}$, a contradiction.

The following result, for ultrafilters, was proved by Leader [2] (Theorem 6) but can easily be extended to ultrafilter bases.

2. Lemma: Every ultrafilter (or ultrafilter base) $\mathcal{U}$ in a proximity space X is a subclass of a unique cluster $\boldsymbol{c}$ from X and $\boldsymbol{c}$ consists of all subsets of X which are near every member of $\mathcal{U}$.

3. Lemma: Let $\boldsymbol{c}$ be a cluster from a proximity space X and $A \in \boldsymbol{c}$. Then there exists an ultrafilter $\mathcal{U}$ which contains A and which is a subclass of $\boldsymbol{c}$.

Proof. Let $\mathcal{U}$ be a maximal collection of subsets of X such that (a) $A \in \mathcal{U}$ (b) $A_i \in \mathcal{U}$ for $i = 1, \ldots, n$ implies $\bigcap_{i=1}^{n} A_i \in \boldsymbol{c}$. Clearly the property of being a member of $\boldsymbol{c}$ satisfies $\boldsymbol{P}$ of Lemma 1, so that $\mathcal{U}$ is an ultrafilter which is a subclass of $\boldsymbol{c}$.

If $\boldsymbol{c}$ and $\mathcal{U}$ are as above, we say that $\boldsymbol{c}$ is *determined by* $\mathcal{U}$ or $\mathcal{U}$ *generates* $\boldsymbol{c}$.

4. Theorem: If A is near B, then there exists a cluster $\boldsymbol{c}$ from X which contains both A and B.

Proof: Let $\mathcal{U}$ be a maximal collection of subsets of X which contains A and such that finite intersections of members of $\mathcal{U}$ are near B. The property of being near B satisfies $\boldsymbol{P}$ of Lemma 1, so that $\mathcal{U}$ is an ultrafilter. The cluster $\boldsymbol{c}$ determined by $\mathcal{U}$ contains both A and B.

5. Theorem: A proximity space X is compact if and only if every cluster from X possesses a point.

Proof: If a cluster $\boldsymbol{c}$ is determined by an ultrafilter $\mathcal{U}$ then $\{x\}$ is in $\boldsymbol{c}$ iff $\mathcal{U}$ converges to x. The theorem then easily follows from Lemmas 2 and 3 by the use of the well-known result: a topological space is compact iff every ultrafilter in it converges.

6. Theorem: If X is a subspace of a proximity space Y, then every cluster $\boldsymbol{b}$ from X is a subclass of a unique cluster $\boldsymbol{c}$ from Y and $\boldsymbol{c}$ consists of all subsets of Y which are near every member of $\boldsymbol{b}$.

Proof: $\boldsymbol{b}$ is determined by an ultrafilter $\mathcal{U}$ in X which is an ultrafilter base in Y. Let $\mathcal{U}$ generate the cluster $\boldsymbol{c}$ in Y. Clearly $\boldsymbol{b}\subset\boldsymbol{c}$ and $\boldsymbol{c}$ is as described in the statement of the theorem. Proof of uniqueness is similar to that of Lemma 2.

7. Theorem: Let $\boldsymbol{c}$ be a cluster from a proximity space Y and let $X\in\boldsymbol{c}$. Then there exists a unique cluster $\boldsymbol{b}$ from X which is a subclass of $\boldsymbol{c}$, and $\boldsymbol{b}=\{A\,|\,A\subset X \text{ and } A\in\boldsymbol{c}\}$.

Proof: By Lemma 3, $\boldsymbol{c}$ is determined by an ultrafilter $\mathcal{U}$ containing X. $\mathcal{U}_X=\{U\cap X\,|\,U\in\mathcal{U}\}$, the trace of $\mathcal{U}$ on X, is an ultrafilter in X and generates a cluster $\boldsymbol{b}$ from X. Clearly $\boldsymbol{b}$ is a subclass of $\boldsymbol{c}$ and has the property mentioned in the statement of the theorem. Uniqueness is proved as usual.

We finally show how the crucial property of the extension $\bar{f}$ of a proximity mapping $f\colon X\to Y$ can be proved (Theorem 5, Leader [1]). If $\boldsymbol{c}$ is a cluster from X, then by Lemma 3 it it determined by an ultrafilter $\mathcal{U}$ in X. $f(\mathcal{U})=\{f(U)\,|\,U\in\mathcal{U}\}$ is an ultrafilter base in Y and generates a cluster $\boldsymbol{d}$ from Y. We let $\bar{f}(\boldsymbol{c})=\boldsymbol{d}$ and show the crucial property of $\boldsymbol{d}$, namely that $\boldsymbol{d}=\{P\,|\,P\subset Y$ and P is near $f(C)$ for every C in $\boldsymbol{c}\}$. If P is near $f(C)$ for every $C\in\boldsymbol{c}$ then P is near $f(U)$ for every $U\in\mathcal{U}$, so that $P\in\boldsymbol{d}$. To prove the converse we note that $f(\boldsymbol{c})\subset\boldsymbol{d}$, since if $C\in\boldsymbol{c}$ then C is near U for each $U\in\mathcal{U}$ and f being a proximity mapping implies $f(C)$ is near $f(U)$ for each $U\in\mathcal{U}$, i. e. $f(C)\in\boldsymbol{d}$. It then follows that $P\in\boldsymbol{d}$ implies P is near $f(C)$ for every $C\in\boldsymbol{c}$.

REFERENCES

[1] S. Leader, *On clusters in proximity spaces*, Fund. Math *47* (1959), 205—213.

[2] S. Leader, *On completion of proximity spaces by local clusters*, Fund. Math. *48* (1960) 201—216.

University of Alberta, Canada.