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A STRUCTURE ON A SET OF SUBSETS

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1. Introduction. The subject of this note is to supply the set of subsets (partitive set) of a given abelian topological group G with an everywhere defined operation $\oplus$ so that $(\mathcal{P}(G), \oplus)$ is abelian semigroup ($\mathcal{P}(G)$ is the set of all subsets of G). In addition $\mathcal{P}(G)$ is supplied by a topology denoted by $\mathcal{T}_{\mathcal{P}}$ so that the operation $\oplus$ is continuous with respect to the topology $\mathcal{T}_{\mathcal{P}}$, and all the possible topological quotient groups of the topological group G are subgroups of the topological semigroup $(\mathcal{P}(G), \oplus, \mathcal{T}_{\mathcal{P}})$.

The inner operation of the abelian group G is $+$, and its topology is $\mathcal{T}$.

All notions used in this note are familiar and could be found, for example, in Pontriagin's book [*P*].

2. Let G be an abelian group and $\{\rho_\alpha ; \alpha \in A\}$, ($A$ index set) a family of equivalence relations in the group G compatible with the operation $+$. For a given $\alpha \in A$, let H^s_α $(s \in S = S(\alpha))$ be an equivalence class with respect to the equivalence relation ρ_α. Let $G_1, G_2 \subset G$ and α be a fixed index. For $g_1 \in G_1$ and $g_2 \in G_2$ denote by $[g_1 + g_2]_\alpha$ the equivalence class with respect to the equivalence relation ρ_α. For different pairs of $(g_1, g_2) \in G_1 \times G_2$ we have (generally speaking) different sets $[g_1 + g_2]_\alpha$. Consider

$$\bigcup_\alpha [g_1 + g_2]_\alpha, \quad (g_1, g_2) \in G_1 \times G_2$$

and $\alpha \in A$ and denote by $G_1 \vee G_2$ the intersection of all equivalence classes with respect to all the possible equivalence relations ρ_α which contain the above union and are compatible with $+$. Now, we can define a binary composition $\oplus$ in $\mathcal{P}(G)$ in the following way:

$$G_1 \oplus G_2 = G_1 \vee G_2$$

Denote by M and N two arbitrary subsets of G.

Proposition 1. *$M \oplus N$ is always an equivalence class for some equivalence relation ρ_α compatible with $\oplus$.*

P r o o f. According to the definition, $M \oplus N$ is the intersection of all epuivalence classes with respect to all the possible equivalence relations which contain the union $\bigcup_{\alpha \in A} [m+n]_\alpha$. It is well-known (cf., for example [*K*]) that there exists an equivalence relation ρ for some family of equivalence relations $\{\rho_\alpha\}$ which is produced or generated by $\bigcup \rho_\alpha$. In the case of all equivalence relations $\{\rho_\alpha\}$ on a set G such a relation ρ must be an element of $\{\rho_\alpha\}$ for some α. The corresponding equivalence class which contains $\bigcup_{\alpha \in A} [m+n]_\alpha$ must be contained in this union.

Corollary. *The operation* $\oplus$ *is associative.*

Proof. Put $K=(M \oplus N) \oplus P$ and $L=M \oplus (N \oplus P)$. According to the Proposition 1., K and L are equivalence classes. Consider, for example, first of them, K. K can be respresented in the form $[m+n+p]_\alpha$ for some $\alpha \in A$ and $m \in M$, $n \in N$ and $p \in P$. But $[m+n+p]_\alpha = [m]_\alpha + [n+p]_\alpha$ and it follows that $[m]_\alpha \supset M$, $[n+p]_\alpha \supset N \oplus P$ and consenquently $K \supset M \oplus (N \oplus P) = L$. By the same arguments we can conclude $L \supset K$, hence $K = L$.

So $(\mathcal{P}(G), \oplus)$ is an abelian semigroup. We call it *Q-semi group of the abelian group G.*

If we consider that part of $\mathcal{P}(G)$ the elements of which are equivalence classes with respect to a fixed equivalence relation ρ_α we obtain a new element of that part by applying the operation $\oplus$. So we have

Proposition 2. *All quotient groups of a given abelian group G are subgroups of the Q-semigroup of the abelian group G.*

Let us now introduce a topology into $\mathcal{P}(G)$.

Let $\mathcal{A}$ be a family of subsets of a given set G and B a subset of G which, generally speaking, does not belong to the family $\mathcal{A}$. We shall denote by $\mathcal{E}_{\mathcal{A}}(B)$ the set consisting of the elements of $\mathcal{A}$ which meet B.

If $\mathcal{A} \cap B = \emptyset$ it follows that $\mathcal{E}_{\mathcal{A}}(B) = \emptyset$.

If $\mathcal{B}$ is a family of sets B, we note

$$\mathcal{E}_{\mathcal{A}}(\mathcal{B}) = \{\mathcal{E}_{\mathcal{A}}(B) \mid B \in \mathcal{B}\}.$$

Taking for $\mathcal{B} = \{O\} = \mathcal{T}$ the family of all open sets of the topological group G and for $\mathcal{A}$ the family whose members are all the equivalence classes with respect to all the possible equivalence relations in G compatible with $+$, and defining a topology of the $\mathcal{P}(G)$ in that way that $\mathcal{E}_{\mathcal{A}}(\mathcal{B})$ is a subbase for that topology, $\mathcal{P}(G)$ becomes topological space. We note it by $(\mathcal{P}(G), \mathcal{T}_{\mathcal{P}})$.

Proposition 3. *The family of sets* $\mathcal{E}_{\mathcal{A}}(\mathcal{T})$ *is a base for the topology* $\mathcal{T}_{\mathcal{P}}$.

Proof. Let $O'_{\mathcal{P}}, O''_{\mathcal{P}} \in \mathcal{E}_{\mathcal{A}}(\mathcal{T})$. $O'_{\mathcal{P}}(O''_{\mathcal{P}})$ contains all the equivalence classes which meet corresponding set $O'(O'')$ in $\mathcal{T}$. Since $\mathcal{T}$ is topology the intersection $O' \cap O''$ is in $\mathcal{T}$ and the set of all equivalence classes which meet $O' \cap O''$ (denote it by O) is in $\mathcal{E}_{\mathcal{A}}(\mathcal{T})$. It is evident that $O_{\mathcal{P}} \subseteq O'_{\mathcal{P}} \cap O''_{\mathcal{P}}$. So the intersection of two elements in $\mathcal{E}_{\mathcal{A}}(\mathcal{T})$ is element of $\mathcal{E}_{\mathcal{A}}(\mathcal{T})$. Since $G \in \mathcal{T}$ we have $\mathcal{P}(G) \in \mathcal{E}_{\mathcal{A}}(\mathcal{T})$ and $\mathcal{E}_{\mathcal{A}}(\mathcal{T})$ is a base for the topology $\mathcal{T}_{\mathcal{P}}$.

Theorem. 1° *Operation* $\oplus$ *is continuous in the topology* $\mathcal{T}_{\mathcal{P}}$.

2° *All the possible topological quotient groups* G/ρ_α *are topological subgroups of* $(\mathcal{P}(G), \oplus, \mathcal{T}_{\mathcal{P}})$.

Proof. 1° Consider an arbitrary neighbourhood $O_{\mathcal{P}}$ of $M \oplus N$ in the topological space $(\mathcal{P}(G), \mathcal{T}_{\mathcal{P}})$. It is enough to take $O_{\mathcal{P}}$ in the form $O_{\mathcal{P}} = \mathcal{E}_{\mathcal{A}}(O)$ for some $O \in \mathcal{T}$. But O is a neighbourhood in G of the element $m+n$ for $m \in M$ and $n \in N$. As G is topological group, there exist neighbourhoods O_m and O_n of the elements m and n respectively, so that $O_m + O_n \subset O \ldots (\Gamma)$.

Put $O^M = \bigcup_{m \in M} O_m$, $O^N = \bigcup_{n \in N} O_n$ where all the possible pairs (O_m, O_n) satisfy (Γ). Let us form $O^M_{\mathcal{P}} = \mathcal{E}_{\mathcal{A}}(O^M)$ and $O^N_{\mathcal{P}} = \mathcal{E}_{\mathcal{A}}(O^N)$. Since for every pair $(m, n) \in M \times N$ the (Γ) is valid, the following relation follows: $O^M + O^N \subset O$. But O^M and O^N are neighbourhoods of M and N respectively and all the equivalence classes of the form $[m+n]_\alpha$ meet O. Consequently $O^M_P \oplus O^N_P \subset O_P$. This implies the continuity of the operation $\oplus$ in the topological space $(\mathcal{P}(G), \mathcal{T}_{\mathcal{P}})$.

2° Consider the map $f\colon G \to \mathcal{P}(G)$ defined in the following way: for $x \in G$, $f(x) = \bigcup_{\alpha \in A} H^x_\alpha$, i.e. the map of $x \in G$ is the union of all equivalence classes with respect to all the possible equivalence relations ρ_α, which contain x. Restriction of f (according to its second coordinate) to the G/ρ_α is nothing else but projection from G onto G/ρ_α. Prove that f is an open mapping. Let $O \in \mathcal{T}$. Then $f(O) = \bigcup_{x \in 0} (\bigcup_{\alpha \in A} H^x_\alpha)$. But this set is open, since it contains all those equivalence classes which meet the open set $O \in \mathcal{T}$ and consequently it represents an element of the base for the topology $\mathcal{T}_{\mathcal{P}}$. It follows that f is an open mapping, and so is the projection itself. But f is also continuous mapping. Let $O_{\mathcal{P}}$ be an open neighbourhood of $f(x)$ $(x \in G,\ f(x) \subset G)$. The neighbourhood $O_{\mathcal{P}}$ is induced by some neighbourhood O of x in G. Consider another neighbourhood O' of x contained in O. Then $f(O') \subset O_{\mathcal{P}}$. Because of: all equivalence classes (with respect to all the possible equivalence relations ρ_α) represented by elements of O' meet O and consequently belong to $O_{\mathcal{P}}$. So f is continuous. The same is the projection of G onto G/ρ_α. As projection is continuous, open and onto, G/ρ_α is a quotient space. It is true for all $\alpha \in A$. The theorem is proved.

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