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CHOICE TOPOLOGY

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0. *Introduction.* Let X be a topological space. The topology on X will be denoted by $\mathcal{T}$ or $\mathcal{T}_X$. If R is an equivalence relation on X, the quotient set obtained by R we shall denote equivalently by $X/_R$ or $D=\{D_\alpha \mid \alpha \in A,\ A$ some index set$\}$. The quotient topology of $X/_R$ will be denoted by $\mathcal{T}/_R$ or $\mathcal{T}_Q$. The topology of the subspace A of X is denoted by $\mathcal{T}/_A$. The set of all subsets of X is $\mathcal{P}(X)$. Recall that the power topology $\mathcal{T}_*$ on $\mathcal{P}(X)$ is defined in [2] in the following way. Subbase for $\mathcal{T}_*$ is the family of sets $\{U_* \mid U \in \mathcal{T}\}$ where $U_*=\{A \mid \emptyset \neq A \subset X$ and $A \cap U \neq \emptyset\}$.

The most natural topology on $X/_R$ is $\mathcal{T}/_R$. Here we introduce another topology on $X/_R$ which is almost as natural as $\mathcal{T}/_R$. It is denoted by $\mathcal{T}_Z$ and called *choice topology*. The way of introducing of $\mathcal{T}_Z$ is as follows. Let $D=\{D_\alpha \mid \alpha \in A\}$. Consider any mapping $\varphi: D \to X$ defined so that $\varphi(D_\alpha) \in D_\alpha$. Such a mapping is called a *choice function*. Let Z denote the family of all choice functions. *$\mathcal{T}_Z$ is the coarsest topology on D for which all choice functions are continuous.*

In this note we examine some properties of $\mathcal{T}_Z$ and its connection with the power topology, obtaining as consequence a theorem of Michael [3] and Franklin [1] (as the corollary of the theorem 1.3).

1. *Some properties*

Theorem 1.1. $\mathcal{T}_Z$ is not coarser than $\mathcal{T}_Q$.

Proof. Consider the mapping $p: X \to D$ defined in the following way: for all $x \in X$ $p(x)$ is the element of D to which x belongs. It is clear that if we have a mapping of a topological space X into the set D and if it is a continuous mapping with respect to some topology $\mathcal{T}_1$ on D the same mapping must be continuous with respect to any other topology on D which is coarser than $\mathcal{T}_1$. To prove that $\mathcal{T}_Z$ is not coarser than $\mathcal{T}_Q$ it is sufficient to prove that from the continuity of p in the topology $\mathcal{T}_Z$ it follows that $\mathcal{T}_Z=\mathcal{T}_Q$. Prove, first, that p is an open mapping. Let O be an open set in X. First of all $p(O) \supseteq \varphi^{-1}(O)$ $(*)$ for some choice function φ. To prove it, consider all those members of D which intersect O. Denote that part by O_D, $O_D = \{A \mid A \in D$, and $A \cap O \neq \emptyset\}$. Consider $O'_D=\{A' \mid A'=A \cap O\}$. According to the axiom of choice there exists a choice function φ for O'_D and consequently also for O_D. The $(*)$ is proved.

From the definition of the topology $\mathcal{T}_Z$ it follows that $\varphi^{-1}(O)$ is open. Suppose that we have $\mathcal{T}_Z \subset \mathcal{T}_Q$. Then p is continuous mapping of $(X, \mathcal{T})$ onto $(D, \mathcal{T}_Z)$ since p is continuous mapping of $(X, \mathcal{T})$ onto $(D, \mathcal{T}_Q)$. Since p is open and continuous mapping of $(X, \mathcal{T})$ onto $(D, \mathcal{T}_Z)$ according to ([2], theorem 3.8.) $\mathcal{T}_Z = \mathcal{T}_Q$, as was to be proved. Let us show by an example that $\mathcal{T}_Z$ could be strictly finer than $\mathcal{T}_Q$. Consider the one dimensional Euclidian space R^1 and take the following partition of $R^1: D_n = (n, n+1)$ and $\bar{D}_n = \{n\}$. Using the same method as in proof of (*) we obtain that $\mathcal{T}_Z$ is discrete. But quotient topology of that set is not discrete since the set $\{n\}$ is not open in R^1. The proof is complete.

Definition. Decomposition D is *I-non-void* if for each $D_\alpha \in D$ there exists some $x \in D_\alpha$ and a neighbourhood $V(x)$ of x, with the property

$$D_\beta \setminus V(x) \neq \emptyset \quad \text{for all} \quad \beta \neq \alpha.$$

As for $\beta = \alpha$, it is possible both $D_\alpha \setminus V \neq \emptyset$ and $D_\alpha \setminus V = \emptyset$.

Theorem 1.2. $\mathcal{T}_Z$ is discrete if and only if the decomposition D is I-non-void.

Proof. Let the decomposition D be I-non-void. From that hypothesis and above definition it follows existence of $x \in D_\alpha$ and $V(x)$. Take the choice function φ which has the following property: $\varphi(D_\beta) \in D_\beta \setminus V(x)$ for $\beta \neq \alpha$ and $\varphi(D_\alpha) \in V(x)$. We have $\varphi^{-1}(V(x)) = D_\alpha$. Hence $\{D_\alpha\}$ as singleton is open in $(D, \mathcal{T}_Z)$. As it is possible deduce for all α, the first part of the statement is proved.

Conversely, suppose that D is not I-non-void. Then for at least one α, all $x \in D_\alpha$ and every $V(x)$ there exists at least one $\beta \neq \alpha$ with the property $D_\beta \subset V(x)$. Then the relation $\{D_\alpha\} = \varphi^{-1}(V(x))$ is not possible, whatever would be φ and $V(x)$, because of the fact that $\varphi^{-1}(V(x))$ contains always at least one point more. It means $\{D_\alpha\} \bar{\in} \mathcal{T}_Z$ and $\mathcal{T}_Z$ is not discrete.

Theorem 1.3. Let f be continuous function on X onto Y, $D = D(f)$ i. e. $D_\alpha = f^{-1}(\alpha)$, $\alpha \in Y$, and let $\bar{f}\colon D \to Y (\bar{f}(D_\alpha) = f(D_\alpha))$ be continuous and open, then

1° $\mathcal{T}_Q = \mathcal{T}_Z$

2° $\mathcal{T}_Q = \mathcal{T}_*^R(X) \,|\, (X \,|\, R)$, where $\mathcal{T}_*^R(X)$ is the family of all R-saturated sets in the $\mathcal{T}$ and consequently in the $\mathcal{T}_*$ (that is $D = \cup R_\alpha$, $\alpha \in A^* \subset A$).

Proof. Prove first that $f^{-1}(O)$ is R-saturated in $\mathcal{T}$ for all $O \in \mathcal{T}_Y$. First of all $f^{-1}(O) \in \mathcal{T}$ since f is continuous. Let for some $x \in D_\alpha$ we have $f(x) \in O$, then we have $f(x) \in O$ for all $x \in D_\alpha$ according to the definition of R.

Consider $\mathcal{G} \in \mathcal{T}_Z$, we have $\mathcal{G}' = \bar{f}(\mathcal{G}) \in \mathcal{T}(Y)$ since $\bar{f}$ is open. From the continuity of f it follows that $f^{-1}(\mathcal{G}') \in \mathcal{T}$ and R-saturated. Set $p^{-1}(\mathcal{G}) = V$, and prove that $V = f^{-1}(\mathcal{G}')$. $f^{-1}(\mathcal{G}') = f^{-1}(\bar{f}(\mathcal{G})) = f^{-1}(\{y \in Y \,|\, y = \bar{f}(D_\alpha), D_\alpha \in \mathcal{G}\})$. If we denote by $|\mathcal{G}|$ the subset of X such that $p(|\mathcal{G}|) = \mathcal{G}$, we have $f^{-1}(\mathcal{G}') = f^{-1}(\{y \in Y \,|\, y \in f(|\mathcal{G}|)\}) = \{x \in X \,|\, f(x) \in \mathcal{G}\} = |\mathcal{G}| = p^{-1}(\mathcal{G})$. Hence p is continuous in $\mathcal{T}_Z$. Having in mind that $\mathcal{T}_Q$ is the finest topology of D for which p is continuous we have $\mathcal{T}_Z \subset \mathcal{T}_Q$. Using it and the theorem 1.1 we obtain 1°.

Consider now the family of all R-saturated sets in the topology $\mathcal{T}_*$, $\mathcal{T}_*^R$, and its subfamily $\mathcal{T}_*^R(X)$ which is isomorphic to the R-saturated sets in $\mathcal{T}$. Let $\mathcal{G} \in \mathcal{T}_Z$. Then for some $V \in \mathcal{T}$ and for all $\varphi \in Z$ we have $\varphi^{-1}(V) = \mathcal{G}$. Let $U \in \mathcal{T}$ be maximal R-saturated set contained in V. Then there exists a choice function φ_0 such that $\varphi_0^{-1}(U) = \mathcal{G}$. [Proof. Let $D_\alpha \in D$ and $D_\alpha \cap V \neq \emptyset$ and $D_\alpha \cap CV \neq \emptyset$. Then there exists a choice function φ_0 having its value in $D_\alpha \setminus V$ and so $\varphi_0^{-1}(V) = \varphi_0^{-1}(U)$ what was to be proved]. So every element of $\mathcal{T}_Z = \mathcal{T}_Q$ is obtained as a projection (being $p(U) = \varphi_0^{-1}(U) = \mathcal{G}$) of a R-saturated set in $\mathcal{T}$. From the isomorphism of $\mathcal{T}_*^R(X) | (X | R)$ and the family of all R-saturated sets in $\mathcal{T}$ it follows 2°.

2. Now we shall establish connection between the power topology on D and the choice topology on D.

Theorem 2.1. Let $\mathcal{T}$ be a topology on X, $\mathcal{T}_*$ the power topology on $\mathcal{P}(X)$, R an equivalence relation on X and $\mathcal{T}'$ an arbitrary topology of X/R. The equality $\mathcal{T}' = \mathcal{T}_* | (X/R)$ is valid if and only if $\mathcal{T}' = \mathcal{T}_Z$.

Corollary (Michael). For any equivalence relation R on X the topology $\mathcal{T}_* | (X/R)$ is finer than the quotient topology $\mathcal{T}/R$.

Proof of the theorem. Let $U_* \in \mathcal{T}/R$. Then $U_* | (X/R) = \{A | \emptyset \neq A \subseteq X, A \in X/R$ and $A \cap U \neq \emptyset$ for some $U \in \mathcal{T}\}$. But for all $\varphi \in Z$ we have $\varphi^{-1}(U)$ is open according to the definition of $\mathcal{T}_Z$ in D. Besides, we have $\varphi^{-1}(U) = \{A | A \in X/R$ and $\varphi(A) \in U\}$. But the fact $\varphi(A) \in U$ for some $\varphi \in Z$ is equivalent to the fact $A \cap U \neq \emptyset$ and consequently we have $\varphi^{-1}(U) = U_* | (X/R)$, i. e. $U_* | (X/R) \in \mathcal{T}_Z$, that is $\mathcal{T}_*/_R \subset \mathcal{T}_Z$.

Conversely, let $V' \in \mathcal{T}_Z$. We have $V'_* = \{A \in X/R | \varphi(A) \in V$ for some $V \in \mathcal{T}$ and some $\varphi \in Z\}$. By the above arguments we obtain $V'_* = \{A | A \cap V \neq \emptyset\} = V_* | (X/R)$ for $V_* \in \mathcal{T}_*$. Consequently $V'_* \in \mathcal{T}_*/_R$, and $\mathcal{T}_Z \subset \mathcal{T}_*/_R$. The proof is complete.

Proof of the corollary. According to the previous theorem $\mathcal{T}_*(X/R) = \mathcal{T}_Z$ but according to the theorem 1.1 $\mathcal{T}_Z \supset \mathcal{T}/_R$ that is the corollary.

3. *Separation properties*

Theorem 3.1. If $(X, \mathcal{T})$ has one of the below listed topological properties, then $(D, \mathcal{T}_Z)$ has the same topological property:

a) T_1,

b) Hausdorff,

c) Regular,

d) Normal.

Proof. a) If $\{x\}$ is closed for all $x \in X$, then there exists $\varphi \in Z$ such that $\varphi^{-1}(x) = D_\alpha$ (where $x \in D_\alpha$). Since φ is continuous, $\{D_\alpha\}$ is closed in D.

The converse is not true: Example. $X = \{a, b, c, d\}$, $\mathcal{T} = \{a, c, ac, X, \emptyset\}$. Decomposition $D = \{D_1, D_2\}$, $D_1 = \{a, b\}$, $D_2 = \{c, d\}$. $\mathcal{T}_Z$ is discrete while $\mathcal{T}$ is not T_1—topology.

b) Let X be Hausdorff-space and $D_\alpha, D_\beta \in D$, $D_\alpha \neq D_\beta$. Then there exist $a \in D_\alpha$ and $b \in D_\beta$, and $a \neq b$. Since X is Hausdorff there exists neighbourhoods $V(a)$ and $V(b)$ of a and b respectively, such that $V(a) \cap V(b) = \emptyset$. But for some choice function $\varphi \in Z$ it is valid $\varphi(D_\alpha) = a$ and $\varphi(D_\beta) = b$, and consequently, $\varphi^{-1}(V(a))$ which is neighbourhood of D_α does not intersect $\varphi^{-1}(V(b))$ (which is neighbourhood of D_β). Hence D is Hausdorff-space.

c) Remark. Regularity in this paper includes that topology is T_1.

Let $D_\alpha \in D$ and F closed subset of D. Denote by $|F|$ union of all equivalence classes D_α which are members of F. Since F is closed there exists a closed set F_1 contained in $|F|$ which intersects all members of F. Let $a \in D_\alpha$. Evidently $a \overline{\in} F_1$. Since $(X, \mathcal{T})$ is regular there exists open V containing a and open W containing F_1 and $V \cap W = \emptyset$. But for some choice function φ we have $\varphi(D_\alpha) = a$ and $\varphi(F) \subset F_1$ so that $\varphi^{-1}(V)$ and $\varphi^{-1}(W)$ are distinct open neighbourhood of D_α and F respectively.

d) Let F_1 and F_2 be two closed sets of $(D, \mathcal{T}_Z)$. Consider subsets of $(X, \mathcal{T})$, $|F_1|$ and $|F_2|$, where $|F| = \bigcup_{\mathcal{G} \in F} \mathcal{G}$. It is evidently $|F_1| \cap |F_2| = \emptyset$. Besides there exists a closed set F' contained in $|F_1|$ and such that $\mathcal{G}_1 \cap F' \neq \emptyset$ for all $\mathcal{G}_1 \in F_1$ or otherwise F_1 would not be closed. In the same way there exists a closed set F'' contained in $|F_2|$ and $F'' \cap \mathcal{G}_2 \neq \emptyset$ for all $\mathcal{G}_2 \in F_2$. Since $(X, \mathcal{T})$ is normal there exist disjoint open sets V_1 and V_2 containing $|F_1|$ and $|F_2|$ respectively. The sets $\varphi^{-1}(V_1)$ and $\varphi^{-1}(V_2)$ are disjoint neighbourhoods of F_1 and F_2; hence $(D, \mathcal{T}_Z)$ is normal.

The theorem is proved.

Obtained results are better than we can conclude on $(D, \mathcal{T}_*/D)$ considered as subspace of $(\mathcal{P}(X), \mathcal{T}_*)$ (see [3], theorem 4.9).

REFERENCES

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