PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 8 (22), 1968. p. 89—92.

A NOTE ON GENERALIZED LAGUERRE POLYNOMIALS

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(Communicated April 17, 1967)

1. In a recent paper [1], the writer has defined the generalized Laguerre polynomials $T_{kn}^{(\alpha)}(x, p)$ by the Rodrigues' formula

$$T_{kn}^{(\alpha)}(x, p) = \frac{1}{n!} x^{-\alpha} e^{px^k} D^n (x^{\alpha+n} e^{-px^k}), \tag{1.1}$$

where k is a natural number. This work of the writer generalizes both the polynomials set of Palas [2] and the general Laguerre polynomials $L_n^{(\alpha)}(x)$. In [1, p. 183] we notice the following operational representation for $T_{kn}^{(\alpha)}(x, p)$:

$$T_{kn}^{(\alpha)}(x, p) = \frac{1}{n!} \prod_{j=1}^{n} (xD - pkx^k + \alpha + j) \cdot 1 \tag{1.2}$$

which can be written in the form

$$T_{kn}^{(\alpha)}(x, p) = \frac{1}{n!} e^{px^k} \prod_{j=1}^{n} (\delta + \alpha + j) e^{-px^k}; \quad (\delta \equiv xD). \tag{1.3}$$

Now let us define the generalized Laguerre polynomials $T_{kn}^{(\alpha)}(x, p)$ by the operational representation (1.3). Using this definition we shall present operational derivation of some results for $T_{kn}^{(\alpha)}(x, p)$. For our purpose we require the formula

$$a^{\delta} f(x) = f(ax). \tag{1.4}$$

Also throughout this paper we shall suppose that expressions of the form $(1+t)^{\delta+\alpha}$ can be formally expanded into an infinite series viz.,

$$\sum_{n=0}^{\infty} t^n \binom{\delta+\alpha}{n}.$$

2. First we note

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,p)\, t^n$$

$$=\sum_{n=0}^{\infty} \frac{t^n}{n!} e^{px^k} \prod_{j=1}^{n} (\delta+\alpha+j)\, e^{-px^k}$$

$$=e^{px^k} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\delta+\alpha+1)_n \cdot e^{-px^k}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$

$$= e^{px^k} (1-t)^{-(\delta+\alpha+1)} \cdot e^{-px^k}$$

$$= (1-t)^{-\alpha-1} e^{px^k} \left(\frac{1}{1-t}\right)^{\delta} \cdot e^{-px^k}$$

$$=(1-t)^{-\alpha-1} e^{px^k} e^{-p\left(\frac{x}{1-t}\right)^k}$$

$$=(1-t)^{-\alpha-1} e^{px^k \{1-(1-t)^{-k}\}}.$$

Thus we have

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,\, p)\, t^n = (1-t)^{-\alpha-1} e^{px^k \{1-(1-t)^{-k}\}}. \tag{2.1}$$

Next we observe

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n)}(x,\, p)\, t^n$$

$$=\sum_{n=0}^{\infty} \frac{t^n}{n!} e^{px^k} \prod_{j=1}^{n} (\delta+\alpha-n+j)\, e^{-px^k}$$

$$=e^{px^k} \sum_{n=0}^{\infty} t^n \binom{\delta+\alpha}{n} \cdot e^{-px^k}$$

$$=e^{px^k} (1+t)^{\delta+\alpha} \cdot e^{-px^k}$$

$$=(1+t)^{\alpha} e^{px^k} (1+t)^{\delta} e^{-px^k}$$

$$=(1+t)^{\alpha} e^{px^k} e^{-p\{x(1+t)\}^k}$$

$$=(1+t)^{\alpha} e^{px^k \{1-(1+t)^k\}}.$$

We have thus proved

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n)}(x,\, p)\, t^n = (1+t)^{\alpha} e^{px^k \{1-(1+t)^k\}}. \tag{2.2}$$

Further we notice

$$\sum_{m=0}^{\infty}\binom{m+n}{n}T_{k(m+n)}^{(\alpha)}(x,p)\,t^m$$

$$=\sum_{m=0}^{\infty}\binom{m+n}{n}\frac{t^n}{(m+n)!}\,e^{px^k}\prod_{j=1}^{m+n}(\delta+\alpha+j)\,e^{-px^k}$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{e^{px^k}}{n!}(\delta+\alpha+n+1)_m\prod_{j=1}^{n}(\delta+\alpha+j)\,e^{-px^k}$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{e^{px^k}}{n!}(\delta+\alpha+n+1)_m\,e^{-px^k}\,n!\,T_{kn}^{(\alpha)}(x,p)$$

$$=e^{px^k}\sum_{m=0}^{\infty}\frac{t^m}{m!}(\delta+\alpha+n+1)_m\,e^{-px^k}\,T_{kn}^{(\alpha)}(x,p)$$

$$=e^{px^k}(1-t)^{-(\delta+\alpha+n+1)}\,e^{-px^k}\,T_{kn}^{(\alpha)}(x,p)$$

$$=(1-t)^{-\alpha-n-1}\,e^{px^k}\left(\frac{1}{1-t}\right)^{\delta}e^{-px^k}\,T_{kn}^{(\alpha)}(x,p)$$

$$=(1-t)^{-\alpha-n-1}\,e^{px^k}\,e^{-p\left(\frac{x}{1-t}\right)^k}\,T_{kn}^{(\alpha)}\left(\frac{x}{1-t},p\right).$$

Thus we have the interesting result

$$\sum_{m=0}^{\infty}\binom{m+n}{n}T_{k(m+n)}^{(\alpha)}(x,p)\,t^m$$

(2.3)

$$=(1-t)^{-\alpha-n-1}\,e^{px^k\{1-(1-t)^{-k}\}}\,T_{kn}^{(\alpha)}\left(\frac{x}{1-t},p\right).$$

Lastly we get

$$\sum_{n=0}^{\infty}\binom{m+n}{m}T_{k(m+n)}^{(\alpha-n)}(x,p)\,t^n$$

$$=\sum_{n=0}^{\infty}\binom{m+n}{m}\frac{t^n}{(m+n)!}\,e^{px^k}\prod_{j=1}^{m+n}(\delta+\alpha-n+j)\,e^{-px^k}$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{m!\,n!}\,e^{px^k}\binom{\delta+\alpha}{n}n!\prod_{j=1}^{m}(\delta+\alpha+j)\,e^{-px^k}$$

$$=\sum_{n=0}^{\infty}t^n\,e^{px^k}\binom{\delta+\alpha}{n}e^{-px^k}\,T_{km}^{(\alpha)}(x,p)$$

$$=e^{px^k}(1+t)^{\delta+\alpha}\,e^{-px^k}\,T_{km}^{(\alpha)}(x,p)$$

$$=(1+t)^{\alpha}\,e^{px^k}\,e^{-p\{x(1+t)\}^k}\,T_{km}^{(\alpha)}(x(1+t),p)$$

$$=(1+t)^{\alpha}\,e^{px^k\{1-(1+t)^k\}}\,T_{km}^{(\alpha)}(x(1+t),p).$$

We have therefore

$$\sum_{n=0}^{\infty} \binom{m+n}{m} T_{k(m+n)}^{(\alpha-n)}(x, p)\, t^n \tag{2.4}$$
$$= (1+t)^{\alpha} e^{p x^k \{1-(1+t)^k\}}\, T_{km}^{(\alpha)}(x(1+t), p).$$

In conclusion we remark that (2.1) was obtained in [1, p. 186], but here we have derived (2.1) in a way which is quite different from the usual method. The technique of this paper has enabled us to derive not only (2.2), but also two other interesting formulas viz., (2.3) and (2.4) which we have deduced at ease. Further it is interesting to note that in particular when $k = p = 1$, the formulas (2.1)—(2.4) reduce to the following corresponding formulas for the general Laguerre polynomials:

$$\sum_{n=0}^{\infty} T_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1} e^{-\frac{xt}{1-t}} \tag{2.5}$$

$$\sum_{n=0}^{\infty} T_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha} e^{-xt} \tag{2.6}$$

$$\sum_{m=0}^{\infty} \binom{m+n}{n} T_{m+n}^{(\alpha)}(x)\, t^m = (1-t)^{-\alpha-n-1} e^{-\frac{xt}{1-t}}\, T_n^{(\alpha)}\left(\frac{x}{1-t}\right) \tag{2.7}$$

$$\sum_{n=0}^{\infty} \binom{m+n}{m} T_{m+n}^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha} e^{-xt}\, T_m^{(\alpha)}(x(1+t)); \tag{2.8}$$

$$T_n^{(\alpha)}(x) \equiv T_n^{(\alpha)}(x, 1) \equiv L_n^{(\alpha)}(x).$$

REFERENCES

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