PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 7 (21), 1967, p. 63–64

ON A CONNEXION BETWEEN LEGENDRE FUNCTIONS

S. S. Pagey

(Communicated April 15, 1966)

1. Fempl, S. (1) has derived the following relation of degrees $\pm\frac{1}{2}$ given by

$$P_{\frac{1}{2}}(x)\,P_{-\frac{1}{2}}(-x)+P_{\frac{1}{2}}(-x)\,P_{-\frac{1}{2}}(x)=4/\pi. \tag{1.1}$$

In this paper we generalise (1.1) and obtain

$$P_{m+\frac{1}{2}}(x)\,P_{m-\frac{1}{2}}(-x)+P_{m+\frac{1}{2}}(-x)\,P_{m-\frac{1}{2}}(x)=\frac{(-1)^m\cdot 4}{(2m+1)\pi}, \tag{1.2}$$

and also derive

$$Q_{m+\frac{1}{2}}(x)\,Q_{m-\frac{1}{2}}(-x)+Q_{m+\frac{1}{2}}(-x)\,Q_{m-\frac{1}{2}}(x)={(-1)^{m+1}\cdot\pi}/{(2m+1)}, \tag{1.3}$$

where m is an integer, $P_\nu(x)$ and $Q_\nu(x)$ are Legendre's functions of first and second kind. The results are believed to be new.

2. Let

$$I=P_{m+\frac{1}{2}}(x)\,P_{m-\frac{1}{2}}(-x)+P_{m+\frac{1}{2}}(-x)\,P_{m-\frac{1}{2}}(x). \tag{2.1}$$

Using [2, p. 140]

$$P_\nu^\mu(-x)=e^{\pm i\nu\pi}\,P_\nu^\mu(x)-\frac{2}{\pi}\,e^{-2\mu\pi}\sin[\pi(\nu+\mu)]\,Q_\nu^\mu(x),$$

we obtain

$$I=(-1)^m\frac{2}{\pi}\left[P_{m+\frac{1}{2}}(x)\,Q_{m-\frac{1}{2}}(x)-P_{m-\frac{1}{2}}(x)\,Q_{m+\frac{1}{2}}(x)\right]. \tag{2.2}$$

From [2, p. 161] we have

$$(1-x^2)\frac{d}{dx}P_\nu^\mu(x)=-\nu x\,P_\nu^\mu(x)+(\nu+\mu)\,P_{\nu-1}^\mu(x), \tag{2.3}$$

and

$$(1-x^2)\frac{d}{dx}Q_\nu^\mu(x)=-\nu x\,Q_\nu^\mu(x)+(\nu+\mu)\,Q_{\nu-1}^\mu(x). \tag{2.4}$$

Hence (2.3) and (2.4) give

$$(1-x^2)\left\{P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x)-Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)\right\}=$$

(2.5)

$$=\nu\left\{P_\nu^\mu(x)\,Q_{\nu-1}^\mu(x)-Q_\nu^\mu(x)\,P_{\nu-1}^\mu(x)\right\}.$$

We have [2, p. 146]

$$(2.6)\qquad (1-x^2)\left[P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x)-Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)\right]=$$

$$=\frac{2^{2\mu}\,\Gamma\left(1+\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}.$$

Hence for $\mu=0$ and $\nu=m+\frac{1}{2}$ using (2.5) and (2.6) in (2.2) we get (1.2).

If we put $m=0$ in (1.2) we get (1.1).

Proceeding on the same lines and using [2, p. 144]

$$Q_\nu^\mu(-x)=-Q_\nu^\mu(x)\cos[\pi(\nu+\mu)]-\frac{1}{2}\pi P_\nu^\mu(x)\cdot\sin[\pi(\nu+\mu)],\ 0<x<1,$$

we get (1.3).

I am greatly indebted to Dr. P. R. Khandekar for his kind help and encouragement during the preparation of this paper.

REFERENCES

[1.] F e m p l, S.: — *On a connexion between Legendres Functions.* Publ. Inst. Math. (Beograd) (N. S.) 1 (15) 81—82 (1962)

[2.] A. E r d é l y i: — *Higher transcendental functions,* Vol. 1. (1953)