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ON MEASURABLE SETS UNDER CERTAIN TRANSFORMATIONS

K. C. Ray

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Let E_n be the n-dimensional Euclidean space of points or vectors $(t_1, t_2 \ldots, t_n)$ t_r real. Let C be any set in E_n and let T denote any general linear transformation of the form:

$$x_i' = \sum_{j=1}^{n} a_{ij} x_j + a_{i,\ n+1}; \quad i = 1, \ldots, n.$$

We shall denote by $T(C)$ the set of points of E_n which are the transformed points of C under T.

In a previous paper [4], among other results, the author has proved the following theorem: Let A and B be two closed bounded sets in E_n having positive measures. Let p be any positive integer. Then there exists a sphere S in E_n such that if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are any vectors in S, then the set of points ξ such that $\xi \in A$ and $\xi + \lambda_r \in B$, $(r = 1, 2, \ldots, p)$ is closed and is of positive measure.

If $T_r (r = 1, 2, \ldots, p)$ denote the transformations of translation, viz., $x_i' = x_i + a_i^r$, $(i = 1, \ldots, n)$, then it is clear that the conclusion of the above theorem depends on showing that the set such as $X = A \cap T_1(B) \cap T_2(B) \cap \ldots \cap T_p(B)$ is a bounded closed set of positive measure.

It is now natural to enquire whether the set X as defined above, still remains a closed set of positive measure if one replaces T_r by a suitable general linear transformation of the form:

$$x_i' = \sum_{j=1}^{n} a_{ij}^r x_j + a_{i,\ n+1}^r; \quad i = 1, \ldots, n.$$

In this paper we have been guided throughout with this consideration and we prove here some theorems which generalise some of the results of [1], [2], [3], [4], [5], [6], [7]. It may be noted, however, that the methods of proofs adopted here differ considerably from the proofs of the corresponding theorems in the particular cases.

In this connection we mention here a well-known result [8, *p.* 162] viz., if C is a Lebesgue measurable set in E_n and T any nonsingular linear transformation in E_n then $T(C)$ is also Lebesgue measurable and $m\{T(C)\} = |D|\, m(C)$, where $|D|$ is the absolute value of the determinant of the transformation and $m(X)$ denotes the Lebesgue measure of X. Throughout the paper we shall understand by sets, the sets of points or vectors in E_n.

Notations: (*i*) Lebesgue measure of any measurable set X will be denoted by $|X|$; (*ii*) by $K(c, \rho)$ we shall understad a sphere in E_n with centre c and radius ρ; (iii) if A and B are two sets, then A/B is the set of points of A which are not in B; (iv) if $S \subset E_n$, then $S' = E_n/S$; (v) the distance set of two sets A and B is the set of all nonnegative numbers $|x-y|$, $x \in A$, $y \in B$; (vi) $A—B$ denotes the difference set of A and B, i.e., the set of all vectors $a-b$ with $a \in A$ and $b \in B$.

Theorem 1. Let A and B be two closed bounded sets having positive measures and p be any positive integer. Then we can find a positive number $M > 0$, $\delta (>0)$ and p vectors $\mu_k = (\alpha^k_{1,\, n+1}, \alpha^k_{2,\, n+1}, \dots, \alpha^k_{n,\, n+1})$ such that if T_k, $(k = 1, 2, \dots, p)$ be any linear transformations given by

$$x'_i = \sum_{j=1}^{u} a^k_{ij} x_j + \alpha^k_{i,\, n+1}; \quad i = 1, \dots, n$$

where

$$\left.\begin{aligned} &(1) \qquad l \leqslant a^k_{ij} < 1 + \frac{\delta}{(M+1)n}, \quad i = j \\ &(2) \qquad 0 \leqslant a^k_{ij} < \frac{\delta}{(M+1)n}, \quad i \neq j \end{aligned}\right\} i, j = 1, \dots, n$$

then the points ξ such that $\xi \in A$ and $T_k^{-1} \xi \in B$, $(k = 1, \dots, p)$ form a closed set of positive measure.

Proof. Since A and B are of positive measures, there exist two closed spheres $K_1 = K(a, r)$ and $K_2 = K(b, s)$, where $s = \left(\frac{p}{p+1}\right)^{\frac{1}{n}} r$ such that $|K_1/A| < \varepsilon |K_1|$ and $|K_2/B| < \varepsilon |k_2|$, where

$$0 < \varepsilon < \frac{1}{2p^2 + 2p + 1} \; [8, \, p. \; 156].$$

Since A and B are bounded, there exists a sphere $K(0, M)$ which contains both A and B. *W*e choose δ such that

$$0 < \delta < \frac{r-s}{2^{n+1} n! \, p^2 (r+1)}.$$

Let T_1^k and T_2^k be the transformations given by

$$\left.\begin{aligned} &T_1^k : \bar{x}_i = x_i + b^k_{i,\, n+1} \\ &T_2^k : x'_i = \sum_{j=1}^{n} a^k_{ij} \bar{x}_j + a^k_{i,\, n+1} \end{aligned}\right\} i = 1, \dots, n; \; k = 1, \dots, p$$

and satisfying (1), (2).

and the vectors $(b^k_{1,\, n+1}, \dots, b^k_{n,\, n+1}) \in K\left(c, \frac{r-s}{2}\right)$ where $a - b = c$. Also, let

$C_k = T_2^k T_1^k (K_2 \cap B)$ and $C = K_1 \cap A$. We shall show that $|X| > 0$, where $X = C \cap C_1 \cap C_2 \cap \dots \cap C_p$.

If x' be the corresponding point of x under T_2^k then from the conditions imposed on the elements a^k_{ij} it follows that $|x - x'| < \delta$. So, $C_k \subset K_1$, $(k = 1, 2, \dots, p)$. We suppose that $T_2^k T_1^k = T_k$, then $C_k = T_k (K_2 \cap B)$.

If D_k be the determinant of T_k, then $D_k > 1 - \sum a_{i_r j_r} a_{i_s j_s} \ldots a_{i_t j_t}$, where the summation contains $\frac{1}{2} n!$ terms.

Again, it can be verified that any term of the above summation satisfies the relation

$$a_{i_r j_r} \ldots a_{i_t j_t} < \left(1 + \frac{\delta}{(M+1)n}\right)^{n-2} \cdot \frac{\delta}{(M+1)n} \cdot \frac{\delta}{(M+1)n}.$$

So, $$D_k > 1 - \frac{n!}{2}\left(1 + \frac{\delta}{(M+1)n}\right)^{n-2} \cdot \frac{\delta^2}{(M+1)^2 n^2}$$
$$> 1 - 2^n n! \, \delta$$
$$> 1 - \frac{r-s}{2p^2(r+1)}$$
$$> 1 - \frac{1}{2p^2}.$$

Now, $|X| \geqslant |K_1| - [|C_1'| + |C_2'| + \cdots + |C_p'| + |C'|$, complements being taken $w.r.t. K_1$. But $|C'| = |K_1/_A|$ and

$$|C_k'| = |K_1| - D_k |K_2| + D_k |K_2/_B| < |K_1| - \left(1 - \frac{1}{2p^2}\right)[|K_2| - |K_2/_B|]$$

So, $$|X| > |K_1| - [p|K_1| - p|K_2| + \frac{1}{2p}|K_2| + \left(p - \frac{1}{2p}\right)|K_2/_B| + |K_1/_A|] =$$
$$= |K_1| - [|K_2| + \frac{1}{2p}|K_2| + \left(p - \frac{1}{2p}\right)|K_2/_B| + |K_1/_A|]$$
$$> K|_1| - \left[|K_2| + \frac{1}{2p}|K_2| + \left(p - \frac{1}{2p}\right)\varepsilon|K_2| + \varepsilon|K_1|\right]$$
$$> 0, \quad \text{since } 0 < \varepsilon < \frac{1}{2p^2 + 2p + 1}.$$

If $\xi \in X$, then $\xi \in K_1 \cap A$ and $\xi \in T_k(K_2 \cap B)$, i. e., $\xi \in A$ and $T_k^{-1}\xi \in B$, $(k = 1, \ldots, p)$. Thus the set of ξ such that $\xi \in A$ and $T_k^{-1}\xi \in B$ is a closed set of positive measure.

Corollary 1. If $a_{ij}^k = 1$, $i = j$ and $a_{ij}^k = 0$, $i \neq j$ for $k = 1, \ldots, p$ then we obtain theorem1 of [4].

Corollary 2. If $A = B, a = b$ and $b_{i,n+1}^k = 0$, $i = 1, \ldots, n$; $k = 1, \ldots p$ then we obtain theorem 1 of [3].

Corollary 3. If $a_{ij}^k = 1$, $i = j$ and $a_{ij}^k = 0$, $i \neq j$ for $k = 1, \ldots, p$ and $A = B$, $a = b$ then we obtain theorem 1 of [1].

If $a_{ij}^k = 1$, $i = j$ and $a_{ij}^k = 0$, $i \neq j$ for $k = 1, .., p$ and $A = B$ but $a \neq b$, we obtain the following

Corollary 4. Let A be a closed bounded set having positive measure. Let p be any positive integer. Then there exists a sphere S (with centre different from the origin) with the following property: if $\lambda_1, \lambda_2, \ldots \lambda_p$ are any vectors in S, then the set of ξ such that $\xi \in A$ and $\xi+\lambda_r \in A$, $r=1,\ldots,p$, is closed and is of positive measure.

Corollary 5. If $n=p=1$ and $a_{ij}^k=1$, $i=j$ and $a_{ij}^k=0$, $i\neq j$ then we obtain a result of [5] which is an extension of a theorem of Steinhaus [7] that the set of distances of two sets with positive measures contains at least one whole interval.

Corollary 6. If A and B are two bounded sets of positive measures, then the difference set $A-B$ contains a sphere [2, 6].

Theorem 2. Let A and B be two closed bounded sets of positive measures. There exist a positive number M and linear transformations

$$T_{\delta_k}: x_i' = \sum_{j=1}^{n} a_{ij}^k x_j + a_{i\,,\,n+1}^k; \quad i=1,\ldots,n; \quad k=1,2,\ldots$$

where the elements a_{ij}^k satisfy the relations (1), (2) replacing δ by δ_k such that if $\{\lambda_k\}$, $\lambda_k>0$ be any null sequence, there exists a subsequence $\{\lambda_{nk}\}$ of $\{\lambda_k\}$ and a point $\xi \in A$ such that $T_{\lambda_{nk}}^{-1}\xi \in B$, $_{k=1,2,\ldots}$

The proof of the theorem follows in the same lines as the proof of Theorem 2 of [4].

Theorem 3. Let A and B be two closed bounded sets of positive measures. There exist a positive number M, a null sequence $\{\mu_r\}$ and vectors $(b_{1,n+1}^r, \ldots b_{n,n+1}^r)$ such that if $\{\delta_r\}$ be any null sequence satisfying

$$0<\delta_r<\min\left\{\mu_r, \frac{1}{2^{n+r+1}\,n!}\right\}$$

and T_{δ_p} is the linear transformation

$$x_i' = \sum_{j=1}^{n} a_{ij}^r x_j + b_{i,n+1}^r; \quad i=1,\ldots,n$$

where

$$\left.\begin{array}{ll} 1 \leqslant a_{ij}^r < 1+\dfrac{\delta_r}{(M+1)n}, & i=j \\ 0 \leqslant a_{ij}^r < \dfrac{\delta_r}{(M+1)n}, & i\neq j \end{array}\right\} i,\ j=1,\ldots,n$$

then the set of points x such that $x\in A$ and $T_{\delta_r}^{-1}x \in B$, $(r=1,\ 2,\ldots)$ forms a closed set of positive measure.

Proof: Since A and B are of positive measures, there exist two closed spheres $K_1=K(a,r)$ and $K_2=K(b,s)$ where $s=\left(\dfrac{p}{p=1}\right)^{\frac{1}{n}} r$ such that $|K/_A|<\varepsilon|K_1|$ and $|K/_B|<\varepsilon|K_2|$, where $0<\varepsilon<\dfrac{1}{2p^2+2p+1}$. Let $A_1=K_1\cap A$ and $B_1= = T_1(K_2\cap B)$, where T_1 is the translation given by $T_1: \bar{x}_i = x_i+\lambda_{i,\,i=1,\ldots,n}$, where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in K\left(c, \dfrac{r-s}{2}\right)$ and $c=a-b$. Then, as in theorem 1, it follows that $A_1\cap B_1$ is a closed set of positive measure. Let $Y=A_1\cap B_1$ be contained in an open sphere $K(0, M)$.

We first show that there exists a positive null sequence $\{\mu_r\}$ such that if $\{\delta_r\}$ be any null sequence satisfying $0<\delta_r<\min\left\{\mu_r, \frac{1}{2^{n+r+1}n!}\right\}$ and $T\delta_p'$ is the linear transformation

$$T'_{\delta_r}: x'_i = \sum_{j=1}^{n} a^r_{ij}\bar{x}_j + a^r_{i,n+1}; \quad i=1,\ldots,n$$

where
$$1 \leqslant a^r_{ij} < 1 + \frac{\delta_r}{(M+1)n}, \quad i=j$$

and
$$0 \leqslant a^r_{ij} < \frac{\delta_r}{(M+1)n}, \quad i \neq j$$

then the set of points x such that $x \in Y$ and $T'^{-1}_{\delta_r} x \in Y, (r=1,2,\ldots)$ forms a closed set of positive measure.

As shown by Kestelman [1], we may define a sequence of open sets $\{U_r\}$ such that $U_1 \supset U_2 \supset U_3 \supset \ldots$:

$$Y = \prod_{r=1}^{\infty} U_r \quad \text{and} \quad \sum_{r=1}^{\infty}\{|U_r| - |Y|\} < \frac{|Y|}{2}.$$

If $\mu_r(>0)$ be the distance between Y and U'_r (Complement of U_r), then obviously $\{\mu_r\}$ is a null sequence of positive numbers. We choose the number $\delta_r(>0)$ such that $\delta_r < \min\left\{\mu_r, \frac{1}{2^{n+r+1}n!}\right\}$ so that $\{\delta_r\}$ is also a positive null sequence. Let $C_r = T'_{\delta_r}(Y)$. If x' of C_r be the transform of a point x of Y under T'_{δ_r}, then it follows from the conditions on the elements a^r_{ij} that $|x'-x|<\delta_r$ $<\mu_r$. So, $C_r \subset U_r$ for every r.

Let $\quad Z = C_1 \cap C_2 \cap \ldots \cap C_r \cap \ldots$

If $\xi \in Z$, then $\xi \in C_r$, for every r, i. e., $T'^{-1}_{\delta_r}\xi \in Y$ for every r and $|\xi - T'^{-1}_{\delta_r}\xi|$ $<\delta_r$. So, $\lim_{r\to\infty} T'^{-1}_{\delta_r}\xi = \xi$. Since Y is closed, so $\xi \in Y$. Hence Z is the set of points ξ of Y such that $T'^{-1}_{\delta_r}\xi \in Y$ for every r. Also since each C_i is closed, Z is closed.

Now $\quad Z = C_1 \cap C_2 \cap C_3 \cap \ldots$

$$= U_1 - \sum_{r=1}^{\infty}(U_1 - C_r)$$

$$= U_1 - \sum_{r=1}^{\infty}(U_1 - U_r + U_r - C_r)$$

$$= Y - \sum_{r=1}^{\infty}(U_r - C_r).$$

Therefore, $|Z| \geqslant |Y| - \sum_{r=1}^{\infty}\{|U_r| - |C_r|\}$.

We can show, as in theorem 1, that if D_{δ_r} denotes the determinant of the transformation T'_{δ_r}, then

$$D_{\delta_r} > 1 - \delta_r (2^n n!)$$

$$> 1 - \frac{1}{2^{r+1}}, \quad r = 1, 2, \ldots$$

So, $|C_r| = D_{\delta_r} |Y| > |Y| - \frac{|Y|}{2^{r+1}}$ and therefore,

$$|U_r| - |C_r| < |U_r| - |Y| + \frac{|Y|}{2^{r+1}}.$$

Therefore, $$|Z| > |Y| - \sum_{r=1}^{\alpha} \left\{ |U_r| - |Y| + \frac{|Y|}{2^{r+1}} \right\} =$$

$$= \frac{|Y|}{2} - \sum_{r=1}^{\alpha} \{ |U_r| - |Y| \} > 0.$$

Now, $\xi \in Z$ implies $\xi \in Y$ and $\xi \in C_r$, i. e., $\xi \in A$ and $\xi \in T'_{\delta_r}(Y)$. But $\xi \in T'_{\delta_r}(Y)$ implies $\overset{-1}{T'_{\delta_r}} \xi \in Y$, i. e., $\overset{-1}{T'_{\delta_r}} \xi \in B_1 = T_1 (K_2 \cap B)$.
Hence, $(T'_{\delta_r} T_1)^{-1} \xi \in B$.

Therefore, $\overset{-1}{T_{\delta_r}} \xi \in B$, where $T'_{\delta_r} T = T_{\delta_r}$.

Thus the set of points $\xi \in A$ and $\overset{-1}{T_{\delta_r}} \xi \in B$ forms a closed set of positive measure. This completes the proof.

Corollary If $A = B$ and $a = b$ we obtain theorem 3 of [3].

Theorem 4. Let $A, A_1, A_2, \ldots, A_{m-1} (m > 1)$ be closed bounded sets having positive measures. Then we can find a positive number M, a number $\delta (> 0)$ and vectors $(a^k_{1, n+1}, \ldots, a^k_{n, n+1})$ such that if T^k_δ be any linear transformation given by

$$T^k_\delta : x'_i = \sum_{j=1}^{n} a_{ij} x_j + a^k_{i, n+1}; \; k = 1, \ldots, m-1 \text{ and satisfying (1), (2),}$$

then the set of points ξ such that $\xi \in A$ and $\overset{-1}{T^k_\delta} \xi \in A_k$, $(k = 1, \ldots, m-1)$ is a closed set of positive measure.

Proof: Since A is a set of positive measure, there exists a closed sphere $\Gamma = K(a, r)$ such $|\Gamma \cap A| > \left(1 - \frac{1}{4(m-1)}\right) \gamma$, where $|\Gamma| = \gamma$. Similarly, there exist closed spheres $\Gamma_k = K(a_k, s)$ such that $|\Gamma_k \cap A_k| > \left(1 - \frac{1}{4(m-1)}\right) \gamma_k$ where $|\Gamma_k| = \gamma_k$, $(k = 1, \ldots, m-1)$ and $s = \left(1 - \frac{1}{2m}\right)^{\frac{1}{n}} r$.

Since the sets are bounded, there exists a sphere $K(0, M)$ which contains all the sets A and A_i, $(i=1, \dots, m-1)$. We choose δ such that

$$0<\delta<\frac{r-s}{2^n n!(r+1)(4m-5)(2m-1)}.$$

Let $c_k=a-a_k$, $k=1, \dots, m-1$. Let the transformations T_1^k and T_2 be given by

$$\left.\begin{array}{l} T_1^k : \bar{x}_i = x_i + c_k,\ k=1, \dots, m-1 \\ T_2 : x_i' = \sum_{j=1}^{n} a_{ij} \bar{x}_j \end{array}\right\} i-1, \dots, n.$$

where
$$\left.\begin{array}{l} 1 \leqslant a_{ij} < 1 + \frac{\delta}{(M+1)n},\ i=j \\ 0 \leqslant a_{ij} < \frac{\delta}{(M+1)n},\ i \neq j \end{array}\right\} i, j=1, \dots, n$$

Let $X=\Gamma \cap A$ and $X_k = T_2 T_1^k (\Gamma_k \cap A_k)$, $k=1, \dots, m-1$. We put $T_2 T_1^k = T_\delta^k$ so that $X_k = T_\delta^k (\Gamma_k \cap A_k)$, $k=1, \dots, m-1$. We show that $|Y|>0$, where $Y = X \cap X_1 \cap X_2 \cap \dots \cap X_{m-1}$.

If x' be the corresponding point of x under T_2 then from the conditions imposed on the elements a_{ij}, it follows that $|x-x'|<\delta$. So, $X_k \subset \Gamma$, $k=1, \dots, m-1$.

If D_δ be the determinant of $T_\delta{}^k$ then, as in theorem 1, we see that

$$\begin{aligned} D_\delta &> 1-2^n n!\,\delta \\ &> 1-\frac{1}{(4m-5)(2m-1)}. \end{aligned}$$

Now, $|Y| > \gamma - [|X'| + |X_1'| + \dots + |X_{m-1}'|]$, where the dashes denote complements $\omega. \gamma. t. \Gamma$.

But $|X'| < \gamma - \left(1-\frac{1}{4(m-1)}\right)\gamma = \frac{1}{4(m-1)}\gamma$

and
$$\begin{aligned} |X_k'| &< \gamma - D_\delta\left(1-\frac{1}{4(m-1)}\right)\gamma_k,\ k=1, \dots, m-1 \\ &= \gamma - D_\delta\left(1-\frac{1}{4(m-1)}\right)\left(1-\frac{1}{2m}\right)\gamma \\ &= \gamma\left[1-D_\delta \frac{(4m-5)(2m-1)}{4(m-1)\cdot 2m}\right]. \end{aligned}$$

Therefore,
$$\begin{aligned} &|X_1'| + \dots + |X_{m-1}'| \\ &< \gamma\left[(m-1) - D_\delta \frac{(4m-5)(2m-1)}{8m}\right] \\ &< \gamma\left[(m-1) - \frac{(4m-5)(2m-1)}{8m} + \frac{1}{8m}\right]. \end{aligned}$$

Hence, $$|Y|>\gamma\left[1-\frac{1}{4(m-1)}-(m-1)+\frac{(4m-5)(2m-1)}{8m}-\frac{1}{8m}\right]$$
$$=\gamma\left[\frac{m^2-2}{4m(m-1)}\right]>0.$$

If $\xi\in Y$, then $\xi\in\Gamma\cap A$ and $\xi\in T_\delta^k(\Gamma_k\cap A_k), k=1,\ldots,m-1$. So, $\xi\in A$ and $\overset{-1}{T_\delta^k}\xi\in A_k$, $k=1,\ldots,m-1$. This proves the theorem.

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Hooghly Institute of Technology, *West Bengal.*

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