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ON AN ENLARGEMENT OF TOPOLOGIES

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To examine a certain property P of a topological space X which is maximal in the meaning that no finer topology possesses the property P, many ways of constructing of new finer topologies are needed. In this note one of the possible constructions is given.

Let (X, τ) be a space. For construction of finer topologies than τ on X we use a family $\sum$ of subsets of X, submitting $\sum$ to be a ring of sets with respect to intersection and symmetric difference operations. Although we mostly use property of $\sum$ to be closed under operation of $\cap$, the assumption of ring structure of $\sum$ is maintained, because that ring structure is a good one in different occasions.

The introduced refinement of the given topology τ is named $\sum$—enlargement of the topology τ. This is its definition.

By $\sum$ —enlargement of the given topology τ we mean a topology τ_Σ the base of which is the following family of subsets of X

$$\tau \cup \{0 \cap S; \quad 0 \in \tau, \quad S \in \sum\}.$$

It is evident that $\tau \subset \tau_\Sigma$.

We use the following notations:

int A means interior of A in the topological space (X, τ); $\mathrm{int}_{\tau_\Sigma} A$ means interior of A in the topology τ_Σ, A^-(resp. $A^{-\tau_\Sigma}$) is closure in the topology τ (resp. τ_Σ).

The use of concepts regular and so on, is according to the J. Kelley's book General Topology, New York, 1955.

The way of introducing the above enlargement is justified by the following theorem.

T$_1$. Let τ and τ' be two topologies on a set X and $\tau \subset \tau'$. There exists a ring $\sum$ such that $\sum$ —enlargement of τ, τ_Σ, is just τ'.

Proof. Put $\sum' = \tau' \backslash \tau$. Let $\sum$ be the ring generated by the family of sets $\sum'$. We shall prove that $\sum$ —enlargement of τ is τ'. Let $\{0_\alpha\}$ be an arbitrary subset of the base of τ_Σ. Then $0_\alpha = 0'_\alpha \cap S_\alpha$ for $0'_\alpha \in \tau$ and $S_\alpha \in \sum$. But all $0'_\alpha \in \tau \subset \tau'$ and all $S_\alpha \in \tau'$, according to the fact that every element of $\sum$ can be covered by a finite union of elements in $\sum' \subset \tau'$ and every such union also belongs to τ'. Hence $0_\alpha \in \tau'$ and $G_\alpha = \cup 0_\alpha \in \tau'$. Unions of such sets G_α are evidently in τ'. If $0_1, 0_2 \in \tau_\Sigma$, we can take $0_1 = 0_1' \cap S_1$, $0_2 = 0_2' \cap S_2$ $0_1', 0_2' \in \tau$ and $S_1, S_2 \in \sum$. $0_1 \cap 0_2 = (0_1' \cap 0_2') \cap (S_1 \cap S_2)$. But being ring $\sum$ is closed under operation of $\cap$, and from the above remark on the structure of $\sum$, it follows $S_1 \cap S_2 \in \tau'$. Hence $\tau_\Sigma \subset \tau'$.

Conversely, all elements of τ' are in $\tau \cup \sum$, and consequently in τ_Σ, so that we have $\tau_\Sigma \supset \tau'$. Hence $\tau_\Sigma = \tau'$.

T$_2$. Let τ_Σ define the enlargement of a topology τ on X and int $(S \cap 0) \neq \varnothing$ for every $S \in \sum$ and every $0 \in \tau$ for which $S \cap 0$ is nonvoid; if τ is a regular topology in which interiors are open, then τ_Σ is also regular.

Proof. Let 0 be open in the topology τ_Σ and $x \in 0$. We shall consider two cases: 1° $0 \in \tau$ and 2° 0 non $\in \tau$.

Case 1°. Since τ is a regular topology and $0 \in \tau$ there exists $0' \in \tau$ such that $x \in 0' \subset (0')^- \subset 0$. Because of $0' \in \tau_\Sigma$ and $\tau \subset \tau_\Sigma$ we have $(0')^{-\tau_\Sigma} \subset (0')'$ and hence $x \in 0' \subset (0')^{-\tau_\Sigma} \subset 0$.

Case 2°. There exists $S \in \sum$ such that for some $0' \in \sum$ we have $x \in 0' \cap S \subset 0$. Since the topology τ is regular, there exists $0'' \in \tau$ such that $x \in 0'' \subset$ int $(0' \cap S)$ and $(0'')^- \subset$ int $(0' \cap S)$.

Taking this into account, and from the fact that $(0'')^{-\tau_\Sigma} \subset (0'')^-$ and int $(0' \cap S) \subset \mathrm{int}_{\tau_\Sigma}(0' \cap S)$ it follows

$$x \in 0'' \subset (0'')^{-\tau_\Sigma} \subset 0' \cap S,$$

i. e. τ_Σ is a regular topology.

T$_3$. If every member of $\sum$ is closed in the topological space (X, τ), then the regularity of τ implies the regularity of τ_Σ.

Proof. Let $x \in 0 \in \tau_\Sigma$. Distinguish as in the case of the proof of the theorem 1, two cases. In the case $0 \in \tau$ the consideration is the same as in the proof above. Consider only the case 0 non $\in \tau$. Then there exist $0 \in \tau$ and $S \in \sum$ such that

$$x \in 0' \cap S \subset 0$$

and because of regularity of the topology τ, there exists $0''$ such that

$$x \in 0'' \subset (0'')^- \subset 0'.$$

Then we have

$$x \in 0'' \cap S \subset (0'' \cap S)^{-\tau_\Sigma} \subset (0'' \cap S)^- \subset (0'')^- \cap S^- = (0'')^- \cap S$$

because S is closed in the topological space (X, τ)
Hence

$$x \in 0'' \cap S \subset (0'' \cap S)^{-\tau_\Sigma} \subset (0'')^- \cap S \subset 0' \cap S \subset 0$$

i. e. τ_Σ is a regular topology.

T$_4$. Let τ be a T_0, T_1 or T_2 topology, then τ_Σ is T_0, T_1 or T_2 topology, respectively.

Proof. Every topology finer than a T_0, T_1 or T_2 topology is also a T_0, T_1 or T_2 topology.

T$_5$. Let $A \subset X$. Topologies τ and τ_Σ induce in A the same topology of subspace if and only if the following condition is fulfilled for all $0 \in \tau_\Sigma$:

$$\mathrm{int}_{\tau_\Sigma}(A \cap 0) \in \tau \qquad (\alpha)$$

Proof. From $\tau \subset \tau_\Sigma$ we have

$$A \cap 0 \setminus \mathrm{int}_{\tau_\Sigma}(A \cap 0) \subset A \cap 0 \setminus \mathrm{int}\,(A \cap 0) \quad (*)$$

If (α) holds the sets on both sides of (*) are the same, hence, both induced topologies on A are identical. If the equality in (*) does not hold for at least one $0_1 \in \tau$ we have that

$$\mathrm{int}\,(A \cap 0_1) = \mathrm{int}_{\tau_\Sigma}(A \cap 0_1)$$

is false, hence, the induced topologies are different.

T$_6$. Let τ_Σ be $\sum$ —enlargement of the topology τ on X and $S \in \sum$. The topologies τ and τ_Σ induce in S the same topology of subspace in one of the two following cases:

1° none of the parts of S belongs to $\sum$;

2° each element of $\sum$, which intersects S, belongs to $\sum$.

T$_7$. Let $\sum$ be a countable σ —ring and (X, τ) a compact topological space. The space (X, τ_Σ) is compact if and only if the set $E = C(\cup S)$, $S \in \sum$, is compact in (X, τ) (CA denotes the complement of A).

Proof. Let (X, τ_Σ) be compact. Since every $S \in \sum$ is open in (X, τ_Σ) the set E is closed and hence compact in (X, τ_Σ). It is also compact in (X, τ) since the topology τ is coarser than τ_Σ.

Conversely, consider an arbitrary cover $\pi = \{0_\alpha ;\ \alpha \in A\}$ of (X, τ_Σ). Every 0 can be represented in the form

$$0'_\alpha \ \cup \ [\bigcup_{\gamma_{\alpha_i}\, \beta_\alpha} (0_{\gamma_\alpha} \cap S_{\beta_\alpha})],\ \gamma_\alpha \in A_\alpha,\ \beta_\alpha \in B_\alpha.$$

Then the family of sets $\{0'_\alpha \cup 0_{\gamma_\alpha}\}$ is a cover of (X, τ). Since (X, τ) is compact, a finite index set $\{\alpha_1, \dots, \alpha_n\} \subset A$ exists with property that the family

$$\pi_1 = \{0'_{\alpha_1} \cup 0_{\gamma_{\alpha_1}}, \dots,\ 0'_{\alpha_n} \cup 0_{\gamma_{\alpha_n}}\}$$

covers X. Let $x \in CE$, then $x \in S_{\beta_\alpha}$ for some $\beta_\alpha \in B_\alpha$, since π covers $X \supset CE$, x must belong to at least one of the sets $0'_{\alpha_i} \cup (0\,\gamma_{\alpha_i} \cap S_{\beta_\alpha})$ for $i = 1, 2, \dots, n$, in virtue of the fact that π_1 is cover of X. It follows that

$$\pi' = \{0_{\alpha_i} \cup [\bigcup_{\gamma_{\alpha_i}, \beta_\alpha} (0_{\gamma_{\alpha_i}} \cap S_{\beta_\alpha})];\ i = 1, \dots, n \text{ covers } CE.$$

Put $S_{\alpha_i} = \bigcup_{\beta_\alpha} S_{\beta_\alpha}$, $\beta_\alpha \in B_\alpha$ where β_α takes only those indices for which

$$S_{\beta_\alpha} \cap 0_{\gamma_{\alpha_i}} \neq \varnothing .$$

The set B_α is at most countable according to the hypothesis of the theorem, and since $\sum$ is σ —ring $S_{\alpha_i} \in \sum$.

Put also

$$0''_{\alpha_i} = \bigcup_{\gamma_{\alpha_i}} 0_{\gamma_{\alpha_i}}$$

Since

$$\bigcup_{\alpha, \beta} (0_\alpha \cap 0_\beta) = (\bigcup_\alpha 0_\alpha) \cap (\bigcup_\beta 0_\beta)$$

is valid in general case, and taking into account the above notation, we have

$$0'_{\alpha_i} \cup [(\bigcup_{\gamma_{\alpha_i}} 0_{\gamma_{\alpha_i}}) \cap (\bigcup_{\beta_\alpha} S_{\beta_\alpha}) = 0'_{\alpha_i} \cup [0''_{\alpha_i} \cap S_{\alpha_i}] \quad i=1,\ldots,n.$$

It follows that

$$\{0'_{\alpha_i} \cup (0''_{\alpha_i} \cap S_{\alpha_i})\}, \quad i=1,\ldots,n \text{ covers } CE.$$

Denote by
$0_1,\ldots,0_n$ elements of that finite cover $(0_i = 0'_{\alpha_i} \cup (0''_{\alpha_i} \cap S_{\alpha_i})$ of CE in τ_Σ.

Of course, $\{0_1,\ldots,0_n\} \subset \pi_1$. From the hypothesis of compactness of E it follows that there exists a subfamily $\{0_{n+1},\ldots,0_m\}$ of π such that $E \subset \bigcup_{i=n+1}^{m} 0_i$, i. e. the finite set $\{0_1\ldots,0_n,\ldots,0_m\}$ of π also covers (X, τ_Σ), as it was to be proved.

T$_8$. A topological space (X, τ) is maximal compact (every topology finer then τ is not compact) if and only if the families of closed and compact sets coincide.

Proof. Suppose there exists a compact not closed set E in the compact space (X, τ). Form $\sum$ —enlargement of τ taking for $\sum$ only one set $E = CE$. Then τ_Σ is compact contrary to the hypothesis that τ is maximal compact topology of X. The converse is easy to prove and so the theorem is valid.