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ABOUT TWIN PARADOX

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In this paper we tried to point at the origins of misunderstanding which led to the problem of twin paradox (called, also, the clock paradox). We showed that there is a physical sense in setting the problem, but that there is no paradox in the matter.

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The special theory of relativity is based on the two axioms:

1. With respect to all inertial systems of reference all physical laws are expressed in the same manner, and
2. the velocity c of light is constant.

This set of axioms is not complete. But, in this paper we shall restrict ourself only to the kinematics. The two mentioned axioms together with the set of axioms of Euclidean three-dimensional geometry represent the complete set of axioms of the kinematics of the special theory of relativity.

The set of axioms of the kinematics of the special theory of relativity differs from the set of axioms of the classical kinematics in acceptance of the two mentioned axioms instead of the classical one: the time is absolute. As regards the fundamental notions and definitions, they are the same in both kinematics, classical and relativistical.

One of the consequences of the axioms of the special theory of relativity is, in contradiction with the classical axiom, that the time is not absolute: the time interval may be measured only with respect to systems of reference, and cannot be measured apart of them. And, in addition, the considered time interval is not the same with respect to all systems of reference.

This characteristic of time seemed to many, by force of habit gained in classical theory and ascribed to experience, very curious one and, therefore, it provokes a fancy.

It is not accidentally that we touched the human psychology in a pure theoretical question. This psychology brought to famous problem of twin paradox. This problem was posed yet in the period of appearance of the special theory of relativity, and the polemics, with respect to this problem, between two completely different points of view takes place even now. To be more precise, there exist two completely different approachements to the mathematical solving of the problem.

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One of consequences of the axiom 1. and 2. are the transformation equations connecting spatial coordinates of two inertial coordinate systems, which belong to two inertial systems of reference, and time coordinates of these systems of reference.

Let us consider two events with same spatial coordinates with respect to one system. Then, according to these equations, the difference of time coordinates — the time interval — of these events with respect to this system is less than the corresponding time interval with respect to the another one. It is usual to express this fact descriptively as „dilatation of time". We shall try to explain the origin of this term. In quest of way of representation of relativistical effects which would be the most acceptable to the spirit familiar to the classical conception of nature, relativists introduced the „observer" at interpretations of relativistic results. Now, let us imagine one observer and one clock moving with respect to this observer, and let us consider two events: at two different instants two different, with respect to the observer, positions of the clock. The time interval between these two events observed by the observer is not equal to the time interval between these two events registered by the clock. Here one always emphasizes that the observer registers „his" time by means of „his" clock, i. e. by means of a clock resting with respect to the observer. In agreement with afore-said, the moving (with respect to the observer) clock registers that a passing time is less than a passing time registered by the observer's clock — the same time interval for the "moving" clock should last "longer" than for the "resting" one. Hence, a "motion" cause a "dilatation" of time.

It is clear that such an interpretation must provoke the fancy: it originated the problem known as twin paradox. This paradox may be stated as follows[1]: "One of the twin brothers stays at home; the other is travelling. When he comes back and meets his brother he is younger than his brother". It would mean that during a mean human life it is possible to reach by rocket some star and come back to the earth although for this travel, according to criterions of classical mechanics, it needs much more time. This would be possible because the "fact" that such a traveller should "move" signifies that his time pass slowlier and, therefore, reduces needed time into limits of his life.

Paradoxical in this story is, at first, that the twin brothers, finishing their job, are not more of the same ages. This paradoxicalness is the classical one. But, the classicist must permit that results of the theory, strange to him, regardless what he means of it, may be in contradiction with those he expects. However, this story seems to be paradoxical also in the realm of the theory of relativity itself. Namely, from the standpoint of the travelling brother (i. e., with respect to his system of reference) he is that who is at rest. He might realise this rest by virtue of a rocket by who he stopped himself, abandoning the earth. And the earth, together with the brother who stayed at home, continued to move. The star, which is at rest with respect to the earth (we imagine it as a fixed star), moves approaching to him till it reaches him. Then, both the star and the earth return, till the earth reaches him and reunites the brothers. After his estimation he is elder than his brother who was staying at home.

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[1] The quoted formulation we transcribed from the Václav Hlavatý's paper; Criticism of the Twin Paradox; Mendelingen van de Koninklijke Vlaamse Academie voor Wetenschappen, Letteren en schone Kunsten van Belgie, 1965, nr 9.

Who is younger between them? After this story it seems that the answer is not something which is characteristic to the nature, but depends on a system of reference with respect to which the problem is solving. This conclusion, in principle, is not uncharacteristic to the theory of relativity. Also, this could be really, i. e. naturally, because, proving experimentally that the classical physics does not design the nature, we proved that the time is not absolute and, therefore, that the age, which is the time interval in fact, is not absolute, also.

However, admitting that the separation between brothers last long enough and that they, with a sufficient velocity, enlarge their distance at first, and then reduce it, we can state the question: what brother will place a wreath on the tomb of the other? The answer would be, in a spirit of our story, that this depends on a system of reference with respect to which we consider the placing of the wreath. Here is a word about two different outcomes and, consequently, it would follow that the outcome existing with respect to one system of reference does not exist with respect to the other. Such conclusion is unreal. Therefore, if this reasoning is in agreement with the theory of relativity, we must conclude that the theory of relativity does not design the nature.

So much attention we have devoted to this discussion because it may be apply, unchangeably, to the phenomenon already observed in reality. It is the word about μ-mesons, those whose whole lives are observed on the earth and those which have arrived to the earth as a component of a cosmic ray. Results concerning them we shall mention later.

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Now, let us consider whether the above reasoning is in agreement with the axioms of the theory of relativity and their consequences. The phenomenon, which we entitled as the "dilatation of time", is the consequence of the transformations of coordinates related with inertial systems of reference. The above reasoning is entirely based upon this "dilatation". However, it is sure that the brothers could not be in rest, both of them, with respect to systems which would be both inertial. All physical reasons at our disposal confirm, in the realm of the special theory of relativity, justifiableness of the assumption that, in the problem of twins, that brother who stays at home is at rest with respect to one system of reference which is inertial with a sufficient approximation, i. e. that the earth is an approximately inertial system of reference.

Relying on the axiom 1. some relativists conclude that, therefore, the interpretation of the problem of twins has not a physical meaning if it assumes the resting of the brother in rocket, because this resting is with respect to a non-inertial system of reference, and that the result based on resting of the brother-host has a physical meaning. Hence, they assert, the relativistic conclusion is that, returning, the brother-traveller will be younger than the brother-host.

In confirmation of their assertion they cite the fact that μ-meson of a cosmic ray succeeds to arrive to the earth, although the necessary time for this is greater than the period of the life experimentally measured for the μ-mesons on the earth. To the question about μ-mesons we shall return once again at the end of this paper.

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Let us return, again, to the twins. The statement that the brother-traveller arriving at home will be younger than his brother-host requires comparison of periods of their lives. The brother-host's period of life is a time interval, measured with respect to the system of reference in which he is at rest, between two events — separation and encountering of the brothers. This is his time. And what is a brother-traveller's period of life? What is his time when there is not an inertial system with respect to which he is at rest?

From this trouble it is possible to extricate oneself as follows. At every instant there exists some inertial system of reference with respect to which the brother-traveler is at rest. For the infinitesimally short period the brother--traveller's time is the time of this system. Integrating these time intervals we get the brother-travaller's time. This time, which is measured by brother-traveller, in literature is known as his proper time τ,

$$\tau = \int_{t_0}^{t} \sqrt{1-\left(\frac{v}{c}\right)^2}\, dt, \tag{1}$$

where t is a time measured with respect to one arbitrary inertial system of reference (for example, with respect to that system with respect to which the brother-host is at rest), and v the brother-traveller's instantaneous velocity with respect to this system.

The proper time τ is an invariant and, therefore, represents the quantity which is characteristic for the brother-traveller. From the definition (1) of the proper time it follows that the brother-host's proper time is just the time of that inertial system of reference with respect to which he is at rest, i.e. the time which we adopted for measuring the period of his life. For this reason it seemed logically to adopt also in the general case of movement (as, for example, in the case of the brother-traveller's movement) the proper time as a measure of a period of time.

Having in mind that the proper times τ_1 and τ_2 of the brother-host and the brother-traveller, respectively, are invariants, it is of no consequence with respect to which inertial system of reference we shall calculate them. Therefore, we shall calculate them with respect to that system with respect to which the brother-host is at rest. If t_0 and $t_1 > t_0$ are instants of separation and of encountering, then

$$\tau_1 = t_1 - t_0 \tag{2}$$

and

$$\tau_2 < t_1 - t_0, \tag{3}$$

because $v \neq 0$. From (2) and (3) it follows

$$\tau_2 < \tau_1, \tag{4}$$

i.e. at the moment of return younger is the brother-traveller.

This consideration may be controvert for two reasons. First, admitting that at every instant the time of that inertial system with respect to which the brother-traveller is in instantaneous rest represents his instantaneuos time, we lose the brother-host's privileged state. We come to the situation in which we could calculate the twins' times with respect to the series of inertial systems with respect to which the brother-traveller is at instantaneous rest. This consideration does not bring into danger the axiom 1., because it takes place in inertial systems. The result of such a consideration may be obtained as follows. The brother-traveller's time τ_2 is the same as in the preceding consideration, because it is defined in the same manner. The brother-host's time τ_1 is given by the formula

$$\tau_1 = \int_0^{\tau_2} \sqrt{1-\left(\frac{v}{c}\right)^2}\, d\tau_2, \tag{5}$$

where v is the brother-host's instantaneous velocity with respect to the system with respect to which the brother-traveller is in an instantaneous rest. On account of $v \neq 0$, it follows from (5) that

$$\tau_1 < \tau_2, \tag{6}$$

i.e. at the moment of encountering the brother-traveller is elder.

The second, more serious remark is that the brother-traveller's proper time is a sum (an integral) of the time intervals measured with respect to the multitude of various inertial systems. Therefore, although each of these infinitesimally short time intervals has a physical meaning (relative, and not absolute), their sum — the proper time — has not it.

This remark releases us from the conclusion (6), but deprives us, also, from the possibility of measurement of brother-traveller's time, and, with this, of an answer to the question: who is younger at the encountering?

There are other attempts to "prove" that the brother-traveller will be younger. For example, in one of them it starts from the relativistic law of addition of velocities. This attempt is based, again, on the set of inertial systems with respect to which the rocket is at instantaneous rest. Correctly assuming that with respect to a such system, for an infinitesimally short time, laws of classical kinematics hold, one tries to come to the required answer. But the error is, evidently, the same as in the preceding case.

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V. Hlavatý in his papers[2] examines the problem. He concludes[3] that from the mathematical point of view there is no paradox, but that, possibly, this problem has not a physical meaning.

Let us investigate the correctness of this conclusion. It is clear that the nature has its answer. It must have the answer to the question: which brother will place a wreath on the tomb of other? This answer is, of course, unknown to us, because the technics does not enable us, up to now, to realize a corresponding experiment. But, has the theory of relativity its answer and what is it?

First of all, we emphasize the fact that the theory of relativity relates a measurement of time only to systems of reference. It does not know some measurement of time which would be related to some body or particle.

Now, let us consider what implies the question: what brother will be younger at the brother-traveller's return? This question signifies that we must

[2] V. Hlavatý: Proper time, apparent time, and formal time in the Twin Paradox, Journal of Mathematics and Mechanics, Vol. 9, № 5, 1960, pp. 733-744, and the paper already cited on the page 68.

[3] Summary of the later paper.

find what courses of time they have spent between two events: separation and encountering. In agreement with the theory of relativity, upon which the time interval is relative, such question has not a physical meaning. In agreement with it this question may be posed only with respect to a given system of reference. But the fact that, then, the answer would be given also with respect to this system, deprives this answer of physical meaning. And, really, the period of life (the time interval) in the theory of relativity is a relative quantity, hence, the quantity which the theory of relativity does not ascribe to the nature. However, the time as a notion (a fundamental notion, as in the classical physics) is absolute in the theory of relativity — the theory of relativity does ascribe it to the nature.

Let us recall the sign of a difference $t_2 - t_1$, where t_1 and t_2 are time coordinates of two events which are separate by the time-like interval. This sign is absolute — it is the same with respect to all inertial systems of reference. The theory of relativity, according to the axiom 1., ascribes it to the nature.

Now, we could pose our task as follows. Let τ_1 and τ_2 be the periods of lives of the brother-host and brother-traveller, respectively, with repect to one whatever inertial system of reference, between their separation and encountering. We ask for the sign of difference $\tau_1 - \tau_2$. If this sign is a function of an inertial system of reference, then, in agreement with Hlavatý's conclusion, the answer *has not* a physical meaning. If this sign is the same with respect to all inertial systems of reference, then the answer *has* a physical meaning. And, the physical meaning would be that the brother-traveller will be younger if this sign is positive.

Hence, let us consider one whatever inertial system of reference S, without defining it at any way nor according to brothers. Let P_1 denote the brother-host's position at any instant t_1 of this system, and P_1' the brother--traveller's position at the same instant. Let P_2 denote the brother-host's position at any other instant t_2, and P_1' the brother-traveller's position at this instant t_2. Then the brother-host's period of life between the events (P_1, t_1) and (P_2, t_2) is

(7) $$\tau_1 = t_2 - t_1,$$

and the brother-traveller's period of life between the events (P_1', t_1) and (P_2', t_2) is

(8) $$\tau_2 = t_2 - t_1.$$

On account that from (7) and (8) $\tau_2 = \tau_1$, i.e.

(9) $$\Delta\tau \equiv \tau_1 - \tau_2 = 0,$$

regardless the positions P_1, P_1', P_2, P_2', it follows that, *with respect to the system* S, the brothers have aged, between these events, by the equal amounts.

Now, we must investigate whether the found result is invariant with respect to transformations of inertial systems.

The time t of some inertial system $\overline{S}$ is determined by the Lorentz transformation

(10) $$\overline{t} = \overline{t}(P, t).$$

It follows that

(11) $$\bar{t}_1 = \bar{t}(P_1, t_1),$$

(12) $$\bar{t}_2 = \bar{t}(P_2, t_2),$$

(13) $$\bar{t}'_1 = \bar{t}(P'_1, t_1),$$

(14) $$\bar{t}'_2 = \bar{t}(P'_2, t_2).$$

Hence, we obtain that, with respect to the system $\bar{S}$, between these two events the brother-host has aged by

(15) $$\bar{\tau}_1 = \bar{t}_2 - \bar{t}_1 = \bar{t}(P_1, t_1) - \bar{t}(P_2, t_2),$$

and the brother-traveller by

(16) $$\bar{\tau}_2 = \bar{t}'_2 - \bar{t}'_1 = \bar{t}(P'_1, t_1) - \bar{t}(P'_2, t_2).$$

Herefrom

(17) $$\Delta\bar{\tau} = \bar{\tau}_1 - \bar{\tau}_2 = \bar{t}(P_1, t_1) - \bar{t}(P'_1, t_1) - [\bar{t}(P_2, t_2) - \bar{t}(P'_2, t_2)].$$

If the first pair of events denotes the brothers' separation (the first "collision"), and the second one their encountering (the second "collision"), then

(18) $$P'_1 \equiv P_1, \quad P'_2 \equiv P_2.$$

Replacing (18) into (17) we get

(19) $$\Delta\bar{\tau} = 0.$$

This confirms the invariability of the result (9), i.e. that the twins, being equally aged at the separation, will be again equally aged at the encountering. This conclusion is, on account to the axiom 1, absolute. And, how much older they will be? — to this question there is not an absolute answer. Any answer to this question is physically meaningless.

We are obliged, yet, to give an interpretation about the phenomenon with μ-mesons. The terms "their real lives" signify nothing. Nor the absolute comparison of their relative lives is possible, for they have only one "collision" — one common event. Therefore, among identities (18) it holds only the second one.