PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 7 (21) 1967, p. 191–195

SOME PERMANENTAL INEQUALITIES

D. Ž. Đoković

(Communicated December 9, 1966)

Notations and preliminaries

S_n is the set of all permutations of $1, \ldots, n$.

$Q_{r,\,n}$ is the set of all strictly increasing sequences of r integers chosen from $1, \ldots, n$.

$G_{r,\,n}$ is the set of all nondecreasing sequences of r integers chosen from $1, \ldots, n$. If $\alpha = (\alpha_1, \ldots, \alpha_r) \in G_{r,\,n}$ and if the distinct indices in $\alpha_1 \leqslant \cdots \leqslant \alpha_r$ are $\beta_1 < \cdots < \beta_k$ $(k \leqslant r)$ then we shall denote by m_i $(i = 1, \ldots, k)$ the multiplicity of β_i in the sequence $\alpha_1, \ldots, \alpha_r$. We define $\mu(\alpha) = m_1! \ldots m_k!$.

$M_{m,\,n}$ is the set of all m by n complex matrices and $M_{n,\,n} = M_n$.

If $\alpha = (\alpha_1, \ldots, \alpha_r) \in G_{r,\,m}$, $\beta = (\beta_1, \ldots, \beta_k) \in G_{k,\,n}$ and $A = (a_{ij}) \in M_{m,\,n}$ then $A[\alpha \mid \beta]$ is r by k matrix whose (s, t) entry is $a_{\alpha_s,\,\beta_t}$ $(s = 1, \ldots, r,\ t = 1, \ldots, k)$.

If $A \in M_{m,\,n}$ then the r^{th} induced matrix $P_r(A)$ of A is $\binom{m+r-1}{r}$ by $\binom{n+r-1}{r}$ matrix whose (α, β) entry $(\alpha \in G_{r,\,m},\ \beta \in G_{r,\,n})$ in the doubly lexicographic ordering is

$$(P_r(A))_{\alpha,\,\beta} = \operatorname{per}(A[\alpha \mid \beta]) / \sqrt{\mu(\alpha)\,\mu(\beta)}.$$

We shall use the following properties of the induced matrix:

(a) $A \in M_{m,\,n}$ & $B \in M_{n,\,k} \Rightarrow P_r(AB) = P_r(A)\,P_r(B)$;

(b) $P_r(A)^* = P_r(A^*)$.

C^n is the unitary space of column n-tuples of complex numbers with respect to the usual inner product.

Let $x_i \in C^n$ $(i = 1, \ldots, k)$ and $X = (x_1, \ldots, x_k) \in M_{n,\,k}$. We define the dot product

$$x_1 \cdot\ \cdots\ \cdot x_k \in C^{\binom{n+k-1}{k}}$$

as the $\binom{n+k-1}{k}$ — tuple whose α^{th} entry $(\alpha \in G_{k,\,n})$ in the lexicographic ordering is

$$\operatorname{per}(X[\alpha \mid 1, \ldots, k]) / \sqrt{\mu(\alpha)}.$$

We have the following property ([2], p. 21):

(c) If $x_i \in C^n$ $(i = 1, \ldots, k)$ and $A \in M_{m,\,n}$ then

$$P_r(A)\, x_1 \cdot\ \cdots\ \cdot x_k = (Ax_1) \cdot\ \cdots\ \cdot (Ax_k).$$

$e_k^{(n)} \in C^n$ n-tuple with 1 in position k, zero elsewhere. If $\alpha=(\alpha_1,\dots,\alpha_r)\in G_{r,n}$ we designate

$$e_\alpha^{(n)} = e_{\alpha_1}^{(n)} \cdot \ \cdots \ \cdot e_{\alpha_r}^{(n)}.$$

$e_\alpha^{(n)}$ has 1 in position α, zero elsewhere. Hence, the vectors $e_\alpha^{(n)} (\alpha \in G_{r,n})$ form an orthonormal basis of $C^{\binom{n+r-1}{r}}$.

We shall need the following results of Marcus and Newman [1]:

(d) If $x_i \in C^n \, (i=1,\dots,k)$ then

$$x_1 \cdot \ \cdots \ \cdot x_k = 0 \Rightarrow x_i = 0 \text{ for some } i=1,\dots,k;$$

(e) If x_i, y_i $(i=1,\dots,k)$ are non-zero vectors in C^n then

$$x_1 \cdot \ \cdots \ \cdot x_k = y_1 \cdot \ \cdots \ \cdot y_k$$

if and only if there exists a set of non-zero scalars $d_1,\dots,d_k$ and a permutation $\sigma \in S_k$ such that $d_1 \dots d_k = 1$ and

$$y_i = d_i x_{\sigma(i)} \qquad i=1,\dots,k.$$

$J \in M_n$ is the matrix all of whose entries are 1.

Ω_n is the set of all doubly stochastic n by n matrices.

We shall need the following result ([2], p. 132):

(f) If $A \in \Omega_n$ is symmetric positive semidefinite then there exists a real symmetric positive semidefinite matrix B such that $B^2 = A$ and $BJ = JB = J$.

Statement of the results

It was proved by Marcus and Newman [1] that

$$|\operatorname{per}(AB)|^2 \leqslant \operatorname{per}(AA^*) \cdot \operatorname{per}(B^*B) \qquad \text{for } A,\ B \in M_n. \tag{1}$$

Our first result is:

Theorem 1. *If $A \in M_{m,n}$, $B \in M_{n,k}$ and $\alpha \in G_{r,m}$, $\beta \in G_{r,k}$ then*

$$|\operatorname{per}((AB)[\alpha|\beta])|^2 \leqslant \operatorname{per}((AA^*)[\alpha|\alpha]) \cdot \operatorname{per}((B^*B)[\beta|\beta]). \tag{2}$$

The equality holds in (2) if and only if one of the following eventualities occurs:

(i) some of the rows $\alpha_1,\dots,\alpha_r$ of A or some of the columns $\beta_1,\dots,\beta_r$ of B are zero;

(ii) no row of A and no column of B mentioned in (i) is zero, and there exists a diagonal matrix D and a permutation matrix P, both in M_r, such that

$$A^*[1,\dots,n|\alpha] = (B[1,\dots,n|\beta])DP.$$

In [1] it is also proved that

$$\operatorname{per} A \geqslant n!/n^n \tag{3}$$

if $A \in \Omega_n$ is symmetric positive semidefinite.

Our second result is:

Theorem 2. *If $A = (a_{ij}) \in \Omega_n$ is symmetric positive semidefinite and $\alpha \in G_{r,n}$ then*

$$\operatorname{per}(A[\alpha|\alpha]) \geqslant r!/n^r \qquad r=1,\dots,n. \tag{4}$$

The equality holds in (4) *if and only if* $A[\alpha|\alpha]$ *has all entries* $1/n$.

Proof of the results

Proof of Theorem 1. On the basis of (*a*), (*b*) and Schwarz inequality we obtain

$$(5)\qquad \frac{|\operatorname{per}((AB)[\alpha|\beta])|^2}{\mu(\alpha)\mu(\beta)} = |(P_r(AB))_{\alpha,\beta}|^2$$
$$= |(P_r(AB)e_\beta^{(k)},\ e_\alpha^{(m)})|^2$$
$$= |(P_r(A)P_r(B)e_\beta^{(k)},\ e_\alpha^{(m)})|^2$$
$$= |(P_r(B)e_\beta^{(k)},\ P_r(A^*)e_\alpha^{(m)})|^2$$
$$\leqslant (P_r(B)e_\beta^{(k)},\ P_r(B)e_\beta^{(k)})(P_r(A^*)e_\alpha^{(m)},\ P_r(A^*)e_\alpha^{(m)})$$
$$= (P_r(B^*B)e_\beta^{(k)},\ e_\beta^{(k)})(P_r(AA^*)e_\alpha^{(m)},\ e_\alpha^{(m)})$$
$$= (P_r(B^*B))_{\beta,\beta}\cdot(P_r(AA^*))_{\alpha,\alpha}$$
$$= \frac{\operatorname{per}((B^*B)[\beta|\beta])}{\mu(\beta)}\cdot\frac{\operatorname{per}((AA^*)[\alpha|\alpha])}{\mu(\alpha)}.$$

It follows that (2) is true.

The equality holds in (5) if and only if $P_r(B)e_\beta^{(k)}$ and $P_r(A^*)e_\alpha^{(m)}$ are linearly dependent. Using (*c*) we get

$$P_r(B)e_\beta^{(k)} = (Be_{\beta_1}^{(k)})\cdot\ \cdots\ \cdot(Be_{\beta_r}^{(k)}),$$
$$P_r(A^*)e_\alpha^{(m)} = (A^*e_{\alpha_1}^{(m)})\cdot\ \cdots\ \cdot(A^*e_{\alpha_r}^{(m)}).$$

Putting

$$(6)\qquad x_i = Be_{\beta_i}^{(k)},\quad y_i = A^*e_{\alpha_i}^{(m)}\qquad i=1,\ldots,r$$

we obtain

$$P_r(B)e_\beta^{(k)} = x_1\cdot\ \cdots\ \cdot x_r,\quad P_r(A^*)e_\alpha^{(m)} = y_1\cdot\ \cdots\ \cdot y_r.$$

These vectors are linearly dependent if and only if

$$(7)\qquad x_1\cdot\ \cdots\ \cdot x_r = 0\ \text{ or }\ y_1\cdot\ \cdots\ \cdot y_r = 0$$

or

$$(8)\qquad x_1\cdot\ \cdots\ \cdot x_r \neq 0,\ y_1\cdot\ \cdots\ \cdot y_r \neq 0\ \text{ and }\ x_1\cdot\ \cdots\ \cdot x_r = dy_1\cdot\ \cdots\ \cdot y_r$$

where d is a non-zero scalar. If (7) is true then (*d*) can be applied and (*i*) follows from (6). If (8) is true then (*e*) can be applied so that

$$y_i = d_i x_{\sigma(i)}\qquad i=1,\ldots,r$$

for appropriate non-zero scalars d_i $(i=1,\ldots,r)$ and a permutation $\sigma\in S_r$. Hence, in this case

$$A^*[1,\ldots,n|\alpha] = (y_1,\ldots,y_r)$$
$$= (d_1x_{\sigma(1)},\ldots,d_rx_{\sigma(r)})$$
$$= (d_{\tau(1)}x_1,\ldots,d_{\tau(r)}x_r)P$$

where $P=(p_{rj})\in M_r$ is permutation matrix such that $p_{\sigma(i),\,i}=1$ $(i=1,\ldots,r)$, $p_{ij}=0$ otherwise, and $\tau=\sigma^{-1}$. With $D=\operatorname{diag}(d_{\tau(1)},\ldots,d_{\tau(r)})\in M_r$ we get

$$A^*[1,\ldots,n\mid\alpha]=(x_1,\ldots,x_r)\,DP=(B[1,\ldots,n\mid\beta])\,DP.$$

The proof is complete.

An example. Taking

$$A=\begin{pmatrix}a_1\cdots a_n\\ b_1\cdots b_n\end{pmatrix},\quad B^T=\begin{pmatrix}c_1\cdots c_n\\ d_1\cdots d_n\end{pmatrix},\quad \alpha=\beta=(1,2),$$

we obtain the following inequality

$$\left|\left(\sum_{i=1}^n a_i c_i\right)\left(\sum_{i=1}^n b_i d_i\right)+\left(\sum_{i=1}^n a_i d_i\right)\left(\sum_{i=1}^n b_i c_i\right)\right|^2$$
$$\leqslant\left[\left(\sum_{i=1}^n |a_i|^2\right)\left(\sum_{i=1}^n |b_i|^2\right)+\left|\sum_{i=1}^n a_i\bar{b}_i\right|^2\right]$$
$$\times\left[\left(\sum_{i=1}^n |c_i|^2\right)\left(\sum_{i=1}^n |d_i|^2\right)+\left|\sum_{i=1}^n \bar{c}_i d_i\right|^2\right].$$

Proof of Theorem 2. Let B be a matrix as in (f). Making use of Theorem 1 we find that

$$\begin{aligned}(r!)^2&=(\operatorname{per}(J[\alpha\mid\alpha]))^2\\&=(\operatorname{per}(BJ)[\alpha\mid\alpha))^2\\&\leqslant\operatorname{per}((BB^*)[\alpha\mid\alpha])\cdot\operatorname{per}(J^*J)[\alpha\mid\alpha])\\&=\operatorname{per}(A[\alpha\mid\alpha])\cdot\operatorname{per}((nJ)[\alpha\mid\alpha])\\&=r!\,n^r\operatorname{per}(A[\alpha\mid\alpha])\end{aligned}\tag{9}$$

which proves (4).

If equality holds in (9) then we must have the case *(ii)* of Theorem 1 for no row of B and no column of J is zero. Therefore we must have

$$\begin{aligned}B[1,\ldots,n\mid\alpha]&=B^*[1,\ldots,n\mid\alpha]\\&=J[1,\ldots,n\mid\alpha]\,DP\\&=J[1,\ldots,n\mid\alpha]\,P(P^{-1}DP)\\&=J[1,\ldots,n\mid\alpha]\,P^{-1}DP\\&=J[1,\ldots,n\mid\alpha]\,D_1\end{aligned}$$

where P is a permutation matrix and D and D_1 nonsingular diagonal matrices. We must have $D_1=I/n$ for all column sums of B are 1. Hence

$$B[1,\ldots,n\mid\alpha]=\frac{1}{n}\,J[1,\ldots,n\mid\alpha].$$

It follows that

$$\begin{aligned} A[\alpha \mid \alpha] &= B[\alpha \mid 1, \ldots, n] B[1, \ldots, n \mid \alpha] \\ &= B^T[\alpha \mid 1, \ldots, n] B[1, \ldots, n \mid \alpha] \\ &= (B[1, \ldots, n \mid \alpha])^T B[1, \ldots, n \mid \alpha] \\ &= \frac{1}{n^2} (J[1, \ldots, n \mid \alpha])^T J[1, \ldots, n \mid \alpha] \\ &= \frac{1}{n^2} J[\alpha \mid 1, \ldots, n] J[1, \ldots, n \mid \alpha] \\ &= \frac{1}{n} J[\alpha \mid \alpha] \end{aligned}$$

which proves the theorem.

Remark. Using the common exterior product and compound matrices we can prove a theorem on determinants analogous to Theorem 1:

Theorem 3. *If* $A \in M_{m,n}$, $B \in M_{n,k}$ *and* $\alpha \in Q_{r,m}$, $\beta \in Q_{r,k}$ *then*

(10) $$|\det((AB)[\alpha \mid \beta])|^2 \leq \det((AA^*)[\alpha \mid \alpha]) \cdot \det((B^*B)[\beta \mid \beta]).$$

The equality holds in (10) *if and only if one of the following eventualities occurs*:

(*i*) *rank* $(A[\alpha \mid 1, \ldots, n]) < r$ *or rank* $(B[1, \ldots, n \mid \beta]) < r$;

(*ii*) *rank* $(A[\alpha \mid 1, \ldots, n]) =$ *rank* $(B[1, \ldots, n \mid \beta]) = r$ *and there exists a nonsingular* $C \in M_r$ *such that*

$$A^*[1, \ldots, n \mid \alpha] = (B[1, \ldots, n \mid \beta]) C.$$

REFERENCES

[1] M. Marcus and M. Newman, *Inequalities for the permanent function* Annals of Mathematics 75 (1962), 47-62.

[2] M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Boston 1964.