PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 7 (21), 1967, p. 185–190

ON THE σ^α SUMMABILITY OF A CLASS OF ASYMPTOTIC SERIES

Branislav Martić

(Communicated December 2, 1966)

1. In [1] *R. P. Agnew* studied the *Lototsky* or $\mathscr{L}$ summability of the classic rapidly divergent series

$$(1.1) \qquad 1-(1!/z)+(2!/z^2)-(3!/z^3)+\cdots$$

and his conclusion is that the series (1.1) is summable $\mathscr{L}$ to

$$V(z)=ze^z\int_z^\infty (e^{-t}/t)\,dt=z\int_0^\infty [e^{-t}/(z+t)]\,dt$$

for each complex number z for which $Re(z)\geqslant\log 2=0{,}69315\ldots.$

This example is particularly interesting as a suggestion of the following general proposition: If the series

$$(1.2) \qquad a_0+(a_1/z)+(a_2/z^2)+(a_3/z^3)+\cdots$$

is the asymptotic expansion of a function $P(z)$ and the coefficient are given by (see [2])

$$(1.3) \qquad a_n=(-1)^{n-1}\int_0^\infty f(t)\,t^{n-1}\,dt \qquad (f(t)>0,\ t>0)$$

for $n=1,2,\ldots,$ what conditions must $f(t)$ satisfy in order that the series (1.2) should be summable by a particular method to $P(z)$ for some finite values of z?

The appropriate summability machinery is found in *Karamata-Stirling's* $KS(\lambda)$ method of summation. In [3] the author proved a quite general theorem regarding this summability. Meanwhile, although $\mathscr{L}$ summability is equivalent to $KS(1)$ summability, the mentioned theorem in [3] does not contain the *Agnew's* result as a special case.

2. It is our purpose to give a better result in these problems. The interesting for this purpose is $\sigma^\alpha\,(\alpha>-1)$ method. We say that $\sum_{n=0}^{\infty} a_n$ is summable $\sigma^\alpha\,(\alpha>-1)$ to s if [4]

$$\sigma^\alpha\{s_n\}=[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^{-1}\sum_{k=0}^{n}\sigma_k^n(\alpha)\,s_k\to s,\ \ n\to\infty$$

where $s_n=\sum_{k=0}^{n} a_k$ and the functions $\sigma_k^n(\alpha)$ are defined by

$$(x+\alpha)(x+\alpha+1)\cdots(x+\alpha+n-1)=\sum_{k=0}^{n}\sigma_k^n(\alpha)\,x^k.$$

The *Lototsky* summability is equivalent to σ^0 summability, i. e. $\mathscr{L}\equiv\sigma^0$.

We shall prove a general theorem which contains *Agnew's* result as a special case.

Theorem. *Let the series* (1.2) *be an asymptotic expansion of*

$$T(z)=a_0+\int_0^{\infty}\frac{f(t)}{z+t}\,dt \qquad (f(t)>0,\ t>0)$$

and let $a_n\,(n=1,2,\ldots)$ *be given by* (1.3). *Assume* $f(t)$ *represents a function* $f(w)$ $(w=u+iv)$ *along the real axis* $u\geqslant 0$ *where* $f(w)$ *is analytic for* $Re(w)\geqslant 0$. *If for every* w *there is a* b *such that*

$$|f(w)|\leqslant C_1 e^{-bu} \qquad (u\geqslant 0,\ b<0)$$

then the series (1.2) *is summable* $\sigma^\alpha\,(\alpha>-1)$ *to* $T(z)$ *for each complex number* z *for which* $Re(z)=x\geqslant b^{-1}\log 2$.

For example, the functions

a) $f(z)=C_1 e^{-bz} \qquad (b>0)$,

b) $f(z)=e^{-bz}(z^k+C_2)^{-1} \qquad (b\wedge C_2>0,\ k=0,1,2\ldots)$,

c) $f(z)=e^{-bz}\prod_{k=1}^{N}(z+C_k) \qquad (b\wedge C_k>0,\ k=1,2,\ldots,N)$

etc., satisfy the condition of Theorem.

Proof of Theorem. Using (1.3) we find for the series (1.2)

$$s_n=\sum_{k=0}^{n}a_k z^{-k}=a_0+\sum_{k=1}^{n}(-1)^{k-1}z^{-1}\int_0^{\infty}f(t)\,(t/z)^{k-1}\,dt=$$

$$=a_0+\int_0^{\infty}\frac{f(t)}{z+t}\,\{1-(-t/z)^n\}\,dt,$$

and hence $\sigma^\alpha\,(\alpha>-1)$ transform of its sequence $s_0, s_1, s_2,\ldots$ of partial sums is

$$\sigma^\alpha\{s_n\}=a_0+\prod_{k=1}^{n}(\alpha+k)^{-1}\sum_{k=0}^{n}\sigma_k^n(\alpha)\left\{\int_0^{\infty}\frac{f(t)}{z+t}\,[1-(-t/z)^k]\,dt\right\}=$$

(2.1)

$$=a_0+\int_0^{\infty}\frac{f(t)}{z+t}\,dt-N_{n-1}$$

where

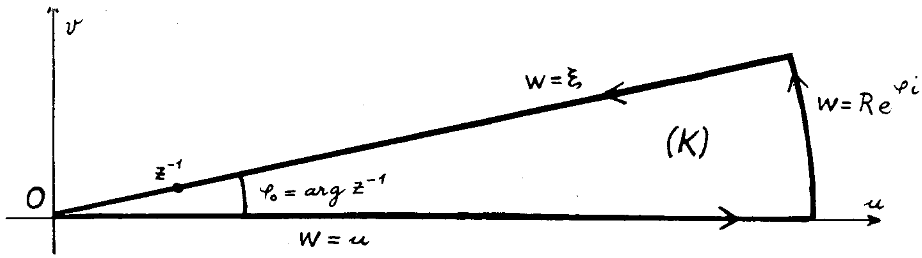
(2.2)
$$N_n=\prod_{k=1}^{n+1}(\alpha+k)^{-1}\sum_{k=0}^{n+1}\sigma_k^{n+1}(\alpha)\int_0^{\infty}\frac{f(t)}{z+t}(-t/z)^k\,dt=$$
$$=\prod_{k=1}^{n+1}(\alpha+k)^{-1}\int_0^{\infty}\frac{f(t)}{z+t}\prod_{k=0}^{n}\left(-\frac{t}{z}+\alpha+k\right)dt.$$

We have to prove that $N_n=0(1)$, $n\to\infty$.

Changing the path of integration in (2.2) by setting $\frac{t}{z}=\xi$ gives

(2.3)
$$N_n=\prod_{k=1}^{n+1}(\alpha+k)^{-1}\int_0^{\infty}\frac{f(z\,\xi)}{1+\xi}\prod_{k=0}^{n}(-\xi+\alpha+k)\,d\,\xi,$$

where the path of integration is the line running from o to ∞ through the point z^{-1}. When $Re(z)=x>0$ we shall show that the path of integration in (2.3) can be replaced by the real line $u\geqslant 0$.



To show this, let (K) be the following closed curve Since $Re(zw)\geqslant 0$ for all points w inside and on (K), it follows that $f(zw)$ is analytic (see Theorem) for all points inside and on (K). Therefore (basic *Cauchy* integral theorem)

$$\oint_{(K)}\frac{f(zw)}{1+w}\prod_{k=0}^{n}(-w+\alpha+k)\,dw=0.$$

We split the last integral into three parts

$$\int_0^{R}\frac{f(zu)}{1+u}\prod_{k=0}^{n}(-u+\alpha+k)\,du+$$
$$+\int_0^{\varphi_0}\frac{f(zRe^{i\varphi})}{1+Re^{i\varphi}}\prod_{k=0}^{n}(-Re^{i\varphi}+\alpha+k)\,iRe^{i\varphi}\,d\varphi+$$
$$\int_{Re^{i\varphi_0}}^{0}\frac{f(z\,\xi)}{1+\xi}\prod_{k=0}^{n}(-\xi+\alpha+k)\,d\,\xi=\tilde{I}_1+\tilde{I}_2+\tilde{I}_3=0,$$

where $\varphi_0 = \arg z^{-1}$. Next

$$|\tilde{I}_2| \leqslant \int_0^{\varphi_0} \frac{|f(zRe^{i\varphi})|}{\sqrt{(1+R\cos\varphi)^2 + R^2\sin^2\varphi}} \prod_{k=0}^{n} (R+|\alpha|+k)\, R d\varphi \leqslant$$

$$\leqslant \int_0^{\varphi_0} |f(zRe^{i\varphi})| \prod_{k=0}^{n} (R+|\alpha|+k)\, d\varphi \leqslant$$

$$\leqslant C_1 \int_0^{\varphi_0} e^{-bR(x\cos\varphi - y\sin\varphi)} \prod_{k=0}^{n} (R+|\alpha|+k)\, d\varphi \leqslant$$

$$\leqslant C_1 \int_0^{\varphi_0} e^{-bR(x\cos\varphi - y\sin\varphi)} (R+|\alpha|+n)^n = 0(1), \quad R \to \infty,$$

since $x\cos\varphi - y\sin\varphi > 0$ when $x>0$ and $0 \leqslant \varphi \leqslant \varphi_0$. Therefore

$$-\tilde{I}_3 = \tilde{I}_1,$$

or

$$\int_0^{\infty} \frac{f(z\xi)}{1+\xi} \prod_{k=0}^{n} (-\xi+\alpha+k)\, d\xi =$$

$$= \int_0^{\infty} \frac{f(zu)}{1+u} \prod_{k=0}^{n} (-u+\alpha+k)\, du,$$

where the path of integration on the left is the line running from o to ∞ through the point z^{-1}, and on the right is the real line $u \geqslant 0$. This and (2.3) gives

$$N_n \doteq \prod_{k=1}^{n+1} (\alpha+k)^{-1} \int_0^{\infty} \frac{f(zu)}{1+u} \prod_{k=0}^{n} (-u+\alpha+k)\, du =$$

$$= \prod_{k=1}^{n+1} (\alpha+k)^{-1} \int_{-\alpha}^{\infty} \frac{f(z(t+\alpha))}{1+\alpha+t} \prod_{k=0}^{n} (-t+k)\, dt =$$

$$= \prod_{k=1}^{n+1} (\alpha+k)^{-1} \left\{ \int_{-\alpha}^{0} + \int_0^{n} + \int_n^{\infty} \right\} =$$

$$= \prod_{k=1}^{n+1} (\alpha+k)^{-1} \{I_1 + I_2 + I_3\}, \tag{2.4}$$

say. In the first place

$$|I_1| \leqslant C_1 e^{-bx\alpha} \int_{-\alpha}^{0} e^{-bxt} \prod_{k=1}^{n} |t-k|\, dt =$$

$$= C_1 e^{-bx\alpha} \int_{0}^{\alpha} e^{bxt} \prod_{k=1}^{n} (t+k)\, dt =$$

(2.5)
$$= O\left\{ \prod_{k=1}^{n} (\alpha+k) \right\}, \quad n \to \infty$$

if $\alpha > 0$. Similar, if $-1 \leqslant \alpha < 0$ setting $\alpha = -a$ $(0 < a < 1)$

$$|I_1| \leqslant C_1 e^{-bx\alpha} \int_{0}^{a} e^{-bxt} \prod_{k=1}^{n} |t-k|\, dt \leqslant$$

(2.6)
$$\leqslant C_1 e^{-bx\alpha}\, n! \int_{0}^{a} e^{-bxt}\, dt = O(n!), \quad n \to \infty.$$

Next

$$|I_2| \leqslant C_1 e^{-bx\alpha} \int_{0}^{n} e^{-bxt} \prod_{k=1}^{n} |t-k|\, dt =$$

(2.7)
$$= C_1 e^{-bx\alpha} \sum_{r=1}^{n} \int_{r-1}^{r} e^{-bxt} \prod_{k=1}^{n} |t-k|\, dt \leqslant$$

$$\leqslant C_1 e^{-ax\alpha}\, n! \sum_{r=1}^{n} \int_{r-1}^{r} e^{-bxt}\, dt = O(n!), \quad n \to \infty$$

since $\prod_{k=1}^{n} |t-k| \leqslant n!$ when $r-1 \leqslant t \leqslant r$ $(r = 1, 2, \ldots, n)$. Finally

$$|I_3| \leqslant C_1 e^{-bx\alpha} \int_{n}^{\infty} e^{-bxt} \prod_{k=1}^{n} |t-k|\, dt =$$

$$C_1 e^{-bx\alpha} e^{-bxn} \int_{0}^{\infty} e^{-bxt} \prod_{k=1}^{n} |t+n-k|\, dt \leqslant$$

$$\leqslant C_1 e^{-bx\alpha} e^{-bxn} \sum_{k=0}^{\infty} \int_{k}^{k+1} e^{-bxk} (n+k)(n+k-1) \cdots (k+1)\, dt =$$

$$= C_1 e^{-bx\alpha} e^{-bxn} n! \sum_{k=0}^{\infty} \binom{n+k}{n} (e^{-bx})^k =$$

$$(2.8) \qquad = C_1 n!\, e^{-bx\alpha} e^{-bxn} (1 - e^{-bx})^{-n-1} =$$

$$= C_1 n!\, e^{-bx\alpha} (1 - e^{-bx})^{-1} (e^{bx} - 1)^{-n}.$$

From (2.4), (2.5), (2.6), (2.7) and (2.8) we obtain

$$| N_n | = \prod_{k=1}^{n+1} (\alpha + k)^{-1} \, O \left\{ \prod_{k=1}^{n} (\alpha + k) + n! + n!\, (e^{bx} - 1)^{-n} \right\} = 0\,(1), \quad n \to \infty$$

for $Re(z) = x \geqslant b^{-1} \log 2$. This and (2.1) complete the proof of Theorem.

REFERENCES

[1] R. P. Agnew, *The Lototsky method for evaluation of series.* Michigan Math. J. 4, (1957), 105-128.

[2] K. Knopp, *Theorie und Anwendung der unendlichen Reihen.* Berlin und Heidelberg, Springer-Verlag (1947).

[3] B. Martić, *O zbirljivosti jedne klase asimptotskih redova postupcima* $\mathcal{K}\, S(\lambda)$. Glas SANU, Od prir-mat nauka (Beograd), CCLXIII, No 28 (1966) 83-91.

[4] V. Vučković, *Eine neue Klasse von Polynomen und ihre Anwendungen in der Theorie der Limitierungsverfahren.* Publ. Inst. Math. Acad. Serbe Sci. 12, (1958), 125-136.