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ON THE SOLUTION OF UNSTEADY LAMINAR BOUNDARY LAYERS PAST BODIES OF REVOLUTION SPINNING ABOUT ITS AXIS

Milan Đ. Đurić

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Abstract. Boundary-layer flows with rotational symmetry are special cases of three-dimensional flow in that they are independent of one of the space coordinates. The method [1] for solution of two-dimensional boundary-layer flows can be extended to these three-dimensional flows. The main feature of the work in this paper lies just in extending the method [1] to above mentioned case, namely, to a particular case — flow past a body of revolution spinning about axis, which is parallel to the stream. At that, as well as in [1], the main-stream flow velocity just outside the boundary-layer is assumed in the form $U(x, t)=V(x)\Omega(t)$, while the angular velocity of the rotation of the body is $\omega(t)$.

1. Introduction. — There are few solved examples of this type of boundary-layers, such as those solved by Illingworth [4] and by Đurić [3]. Illingworth has solved the case of flow past a body of revolution spinning about its axis, which is parallel to the stream; the whole flow being started impulsively from rest, namely, the case $\omega(t)=\omega_0$ and $U(x, t)=V(x)$, while Đurić has solved the case of power and exponential law of change of velocities $\omega(t)$ and $U(x, t)$ in time.

Following the paper [1], we also introduce here new variables instead of old ones, to solve in the abstract mentioned problem. New variables are those ones introduced in [1], namely

$$x, \quad \tau=\frac{1}{\nu}\int_0^t \Omega^2(t)\,dt, \quad \eta=\frac{\Omega(t)}{\nu\sqrt{3\tau}}\,y.$$

By introducing new variables, solutions of this problem can be given in the form of series expansions with coefficients depending on η and τ. For finding these coefficients we obtain simultaneous systems of partial equations. Solutions of these partial equations are given in the form of power seıies. As, in the paper [1] we have studied in details two classes of problems i.e. classes (4.2) from great classes (5.2), known as cases I and II, here, we study in details only the class which corresponds to case I in [1]. But, it is a very simple matter to study also in details a class corresponding to case II in [1], and also a sequence of other cases which can be solved by this method and whose classification is done below.

2. Basic equations and their transformations. — The boundary-layer equations for the case of unsteady incompressible flow past a body of revolution spinning about its axis, which is parallel to the stream are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{w^2}{r}\frac{dr}{dx}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{r}\frac{dr}{dx}=\nu\frac{\partial^2 w}{\partial y^2}, \tag{2.2}$$

$$\frac{\partial (ru)}{\partial x}+\frac{\partial (rv)}{\partial y}=0. \tag{2.3}$$

The boundary conditions being

$$\begin{array}{llll} u=U(x,\ t), & v=0, \quad w=r\,\omega(t), & y=0 \text{ and } t=0, \\ u=v=0, & w=r\,\omega(t), & y=0 \text{ and } t>0, \\ u=U(x,\ t), & w=0, & y=\infty. \end{array} \tag{2.4}$$

In above equations x, y are the distances along and normal to the wall and z is the distance along the arc of a transversal crossection; $u(x, y, t)$, $v(x, y, t)$ and $w(x, y, t)$ are the components of velocity in the direction of x, y and z respectively; ν is the constant kinematic viscosity; t is the time; $r(x)$ is the radius of transversal crossection; $U(x, t)=V(x)\,\Omega(t)$ is the main-stream velocity and $\omega(t)$ is the angular velocity.

In the paper [1] assuming that the external velocity distribution is given as $U(x, t)=V(x)\,\Omega(t)$, and that the function $\Omega(t)$ is defined in any t from $[0, \infty)$, and its square integrable in some interval $[0, t]$, the new variables are defined

$$x, \quad \tau=\frac{1}{\nu}\int_0^t \Omega^2(t)\,dt, \quad \eta=\frac{\Omega\,y}{\nu\sqrt{3\tau}}, \tag{2.5}$$

and

$$\mathfrak{F}(x,\ \eta,\ \tau)=\frac{\psi(x,\ y,\ t)}{\nu\sqrt{3\tau}\;V(x)}, \tag{2.6}$$

for every $x>0$, $y\geqslant 0$ and $t>0$. If, to the above dependent variable (2.6) we add still a variable

$$\mathfrak{R}(x,\ \eta,\ \tau)=\frac{w(x,\ y,\ t)}{r(x)\,\omega(\mathrm{t})}, \tag{2.7}$$

then we shall have solutions of boundary-layer flows past bodies of revolution spinning about its axis, defined by (2.1) — (2.4).

Introducing the stream function $\psi(x, y, t)$ into (2.1) — (2.3) by

$$u=\psi_y, \quad v=-\psi_x-\frac{1}{r}r'\,\psi, \tag{2.8}$$

the equation (2.3) will be identically satisfied and equations (2.1) and (2.2) reduced to the new form. By substitution (2.5) — (2.7) in this manner obtained equations, we obtain

$$\begin{aligned}\mathfrak{F}_{\eta\eta\eta} + \alpha(\tau)(1-\mathfrak{F}_\eta - \eta\,\mathfrak{F}_{\eta\eta}) + \frac{3}{2}\eta\,\mathfrak{F}_{\eta\eta} - 3\tau\,\mathfrak{F}_{\eta\tau} + \beta(\tau)\Big\{V'(1-\mathfrak{F}_\eta^2 + \mathfrak{F}\mathfrak{F}_{\eta\eta}) + \\ + V(\mathfrak{F}_x\,\mathfrak{F}_{\eta\eta} - \mathfrak{F}_\eta\,\mathfrak{F}_{x\eta}) + \frac{r'}{r}V\mathfrak{F}\mathfrak{F}_{\eta\eta}\Big\} + \gamma(\tau)\frac{r'r}{V}\mathfrak{R}^2 = 0,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathfrak{R}_{\eta\eta} + \left(\frac{3}{2} - \alpha(\tau)\right)\eta\,\mathfrak{R}_\eta - 3\tau\,\mathfrak{R}_\tau - \delta(\tau)\,\mathfrak{R} + \beta(\tau)\Big\{V\frac{r'}{r}(\mathfrak{F}\mathfrak{R}_\eta - 2\,\mathfrak{F}_\eta\,\mathfrak{R}) + \\ + V(\mathfrak{R}_\eta\,\mathfrak{F}_x - \mathfrak{F}_\eta\,\mathfrak{R}_x) + V'\mathfrak{F}\mathfrak{R}_\eta\Big\} = 0.\end{aligned} \tag{2.10}$$

where following notations are indroduced

$$\alpha(\tau) = \frac{\dot{\Omega}\,\nu\,3\tau}{\Omega^3}, \quad \beta(\tau) = \frac{\nu\,3\tau}{\Omega}, \quad \delta(\tau) = \frac{\dot{\omega}\,\nu\,3\tau}{\omega\,\Omega^2}, \quad \gamma(\tau) = \frac{\omega^2\nu\,3\tau}{\Omega^3}. \tag{2.11}$$

The boundary conditions being

$$\begin{aligned} &\mathfrak{F}_\eta(x,\ 0,\ \tau) = 1 && \mathfrak{R}(x,\ 0,\ \tau) = 1, && t = 0, \\ &\mathfrak{F}(x,\ 0,\ \tau) = \mathfrak{F}_\eta(x,\ 0,\ \tau) = 0, && \mathfrak{R}(x,\ 0,\ \tau) = 1, && t > 0 \\ &\lim_{\eta\to\infty}\mathfrak{F}_\eta(x,\ \eta,\ \tau) = 1, && \lim_{\eta\to\infty}\mathfrak{R}(x,\ \eta,\ \tau) = 0. && \end{aligned} \tag{2.12}$$

Assuming that functions $V(x)$ and $r(x)$ are those ones of the class C^n $(n = 0, 1, 2, \ldots)$ and continuous, then solutions of partial equations (2.9) and (2.10) can be found in the form of following series expansions

$$\mathfrak{F}(x,\ \eta,\ \tau) = \mathfrak{F}_0(\eta,\ \tau) + V'\,\mathfrak{F}_1(\eta,\ \tau) + V\frac{r'}{r}\,\mathfrak{F}_{1a}(\eta,\ \tau) + \frac{rr'}{V}\,\overset{*}{\mathfrak{F}}_1(\eta,\ \tau) + \cdots, \tag{2.13}$$

$$\mathfrak{R}(x,\ \eta,\ \tau) = \mathfrak{R}_0(\eta,\ \tau) + V'\,\mathfrak{R}_1(\eta,\ \tau) + V\frac{r'}{r}\,\mathfrak{R}_{1a}(\eta,\ \tau) + \cdots \tag{2.14}$$

Comparing the solution (2.13) from here with the solution (2.11) from [1], we see that there are more terms in the series expansion (2.13) than in (2.11) from [1]. Stopping at the second approximation in (2.13) we obtain an extension of two terms relating to (2.11) from [1].

By substitution of (2.13) and (2.14) into (2.9) and (2.10) we obtain two simultaneous systems of partial equations. The first system is

$$\begin{aligned} &L(\mathfrak{F}_0) = -\alpha(\tau), \\ &L(\mathfrak{F}_1) = -\beta(\tau)(1 - \mathfrak{F}_{0\eta}^2 + \mathfrak{F}_0\,\mathfrak{F}_{0\eta\eta}), \\ &L(\mathfrak{F}_{1a}) = -\beta(\tau)\,\mathfrak{F}_0\,\mathfrak{F}_{0\eta\eta}, \\ &L(\overset{*}{\mathfrak{F}}_1) = -\gamma(\tau)\,\mathfrak{R}_0^2, \end{aligned} \tag{2.15}$$

where

$$L=\frac{\partial^3}{\partial\eta^3}-\alpha(\tau)\left(\frac{\partial}{\partial\eta}+\eta\frac{\partial^2}{\partial\eta^2}\right)+\frac{3}{2}\eta\frac{\partial^2}{\partial\eta^2}-3\tau\frac{\partial^2}{\partial\eta\,\partial\tau} \tag{2.15'}$$

is a linear partial operator with variable coefficients. Introducing also the same type operator

$$K=\frac{\partial^2}{\partial\eta^2}-\alpha(\tau)\eta\frac{\partial}{\partial\eta}+\frac{3}{2}\eta\frac{\partial}{\partial\eta}-3\tau\frac{\partial}{\partial\tau}-\delta(\tau), \tag{2.16'}$$

the second system can be written as

$$\begin{aligned} &K(\mathfrak{R}_0)=0,\\ &K(\mathfrak{R}_1)=-\beta(\tau)\,\mathfrak{F}_0\,\mathfrak{R}_{0\eta},\\ &K(\mathfrak{R}_{1a})=-\beta(\tau)\,(\mathfrak{F}_0\,\mathfrak{R}_{0\eta}-2\,\mathfrak{F}_{0\eta}\,\mathfrak{R}_0). \end{aligned} \tag{2.16}$$

The first two equations in the system (2.15) for $\mathfrak{F}_0(\eta,\tau)$ and $\mathfrak{F}_1(\eta,\tau)$ are the same as those ones in [1], formula (2.13), while $\mathfrak{F}_{1a}(\eta,\tau)$ and $\overset{*}{\mathfrak{F}}_1(\eta,\tau)$ represent an extension, in the second approximation for $\mathfrak{F}(x,\eta,\tau)$, relating to a two-dimensional problem considered in [1]. But, the first two equations in the system (2.16) are the same as those ones in case of thermal boundary-layers, as $\sigma=1$ there (see (2.21) in [2]). The boundary conditions are

$$\begin{aligned} &\mathfrak{F}_0=\mathfrak{F}_{0\eta}=0, &&\mathfrak{F}_1=\mathfrak{F}_{1\eta}=0,\ldots, &&y=0,\\ &\mathfrak{F}_{0\eta}\to 1, &&\mathfrak{F}_{1\eta}\to 0,\ldots, &&y\to\infty, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\mathfrak{R}_0=1, &&\mathfrak{R}_1=\mathfrak{R}_{1a}=\cdots=0, &&y=0,\\ &\mathfrak{R}_0\to 0, &&\mathfrak{R}_1\to 0,\ldots, &&y\to\infty. \end{aligned} \tag{2.18}$$

Because of rotation in the set of principal functions still two functions appear in comparison to a two-dimensional case and also to an axi-symmetrical case. These functions are

$$\delta(\tau)=\frac{\dot{\omega}\,\nu\,3\tau}{\omega\,\Omega^2}, \qquad \gamma(\tau)=\frac{\omega^2\,\nu\,3\tau}{\Omega^3}. \tag{2.19}$$

The function $\delta(\tau)$ is the same as the thermal principal function in [2], only instead of $\theta(t)$, here, ought to stand $\omega(t)$.

The only explicit entry of the angular velocity in equations (2.15) and (2.16) is still through these functions. We notice that the function $\delta(\tau)$ is similar to $\alpha(\tau)$, and because of their essential importance we shall name them *the components of a local principal function* in x and z direction respectively. In the same way as $\alpha(\tau)$ also $\delta(\tau)$ is to be written in the form

$$\delta(\tau)=3\tau\frac{d}{d\tau}\ln\frac{\omega}{\omega_0}. \tag{2.20}$$

It also can be expressed as a local shape parameter

$$\delta(\tau)=\frac{1}{\eta_0^2}\frac{\delta_x^{*2}\dot{\omega}}{\nu\omega}, \tag{2.21}$$

analogously to $\alpha(\tau)$. In (2.21) δ_x^* is the component of a displacement thickness in x direction

$$\delta_x^* = \frac{\nu\sqrt{3\tau}}{\Omega}\eta_0, \quad \text{where} \quad \eta_0 = \lim_{\eta\to\infty}(\eta - \mathfrak{F}). \tag{2.22}$$

In the same direction the transformed component of momentum thickness is

$$\delta_x^{**} = \frac{\nu\sqrt{3\tau}}{\Omega}\int_0^\infty \mathfrak{F}_\eta(1-\mathfrak{F}_\eta)\,d\eta. \tag{2.23}$$

The same magnitudes but in z direction have similar form as those ones written above, namely

$$\begin{aligned}\delta_z^* &= \frac{\nu\sqrt{3\tau}}{\Omega}\eta_1, \quad \text{where} \quad \eta_1 = \lim\left(\eta - \int_0^\eta \mathfrak{R}\,d\eta\right),\\ \delta_z^{**} &= \frac{\nu\sqrt{3\tau}}{\Omega}\int_0^\infty \mathfrak{R}(1-\mathfrak{R})\,d\eta.\end{aligned} \tag{2.24}$$

For the skin friction in x respectively z direction, from

$$N_x = \mu\left(\frac{\partial u}{\partial y}\right)_0, \quad N_z = \mu\left(\frac{\partial w}{\partial y}\right)_0,$$

we obtain the following formulae

$$\begin{aligned}\frac{1}{\rho}N_x &= U(x,\ t)\frac{\Omega(t)}{\sqrt{3\tau}}\mathfrak{F}_{\eta\eta}(x,\ 0,\ \tau),\\ \frac{1}{\rho}N_z &= r(x)\,\omega(t)\frac{\Omega(t)}{\sqrt{3\tau}}\mathfrak{R}_\eta(x,\ 0,\ \tau).\end{aligned} \tag{2.25}$$

In the case of axi-symmetrical flows past a body of revolution (without a rotation of the body) z-components of boundary-layer magnitudes are identically equal to zero.

From (2.20) we are able to find a relation $\omega = \omega(\tau)$. It follows

$$\omega(\tau) = \omega_0 \exp\left(\frac{1}{3}\int_{\tau_0}^{\tau}\frac{\delta(\tau)}{\tau}\,d\tau\right), \tag{2.26}$$

namely, we find ω as a unique function of τ respectively t because of a unique relation between τ and t. In application, the function $\omega(t)$ is usually prescribed, and ought to find principal functions. We shall just consider this normal case. Namely, we shall assume, in future, that the function $\omega(t)$ is given as a continuous function and that it belongs to the class C^m $(m = 0, 1, \ldots)$.

3. A class of problems and its generalization. — We shall observe here only a class of problems given by $\omega(t)$. Let us assume that the continuous function $\omega(t)$ belongs to the class C^{∞} and that it can be expanded in a series

$$\omega(t)=\sum_{i=0}^{\infty}\omega_i t^i, \tag{3.1}$$

with $\omega_0 \neq 0$, and otherwise all ω_k are arbitrary. If the external velocity distribution is given by (4.1) in [1], known as case I, then for functions (2.19) we obtain

$$\begin{aligned}\delta(\tau)&=\sum_{i=0}^{\infty}\delta_i\tau^i, \quad \text{with} \quad \delta_0=0,\\ \gamma(\tau)&=\sum_{k=1}^{\infty}\gamma_k\tau^k.\end{aligned} \tag{3.2}$$

Now, from (2.19) substituting (4.1) from [1] and (3.1) instead of functions $\Omega(t)$ and $\omega(t)$ we find coefficients δ_i and γ_i. Since, the principal function $\delta(\tau)$ has the same form as that one in case of thermal-layers (see (2.9) from [2]), only instead of $\theta(t)$ ought to stand $\omega(t)$, then also coefficients δ_i have formally the same form as those ones in [2]

$$\begin{aligned}\delta_0&=0\\ \delta_1&=3c_1\\ \delta_2&=-3c_1^2-6c_1d_1+6c_2\end{aligned} \tag{3.3}$$

but here it is

$$c_i=\frac{\nu^i\omega_i}{\Omega_0^{2i}\omega_0}, \quad d_i=\frac{\nu_i\Omega_i}{\Omega_0^{2i+1}}.$$

The coefficients γ_i are

$$\begin{aligned}\gamma_0&=0\\ \gamma_1&=3e_1\\ \gamma_2&=-9e_1+6g_1\\ \gamma_3&=27e_1^2/e_0-24e_1g_1/e_0-9e_2+3g_1^2/e_0+6g_2\end{aligned} \tag{3.4}$$

where

$$e_i=\frac{\nu^{i+1}\omega_0^2\Omega_i}{\Omega_0^{2i+4}}, \quad g_i=\frac{\nu^{i+1}\omega_0\omega_i}{\Omega_0^{2i+3}}.$$

As in [1], also here, we can introduce non-dimensional magnitudes and show that above systems of formulae remain formally the same.

Now, we can make the generalization of the given problem. Namely, we can put a question, to which angular velocities belongs the component $\delta(\tau)$ of a local principal function given by

$$\delta(\tau)=\delta_0+\delta_1\tau+\delta_2\tau^2+\cdots \tag{3.5}$$

Because of similarity of the function $\delta(\tau)$ here, and that one in [2], following the work in 2, we can immediately write

$$\omega(t)=t^n(k_0+k_1t^{2m+1}+\cdots), \tag{3.6}$$

where

$$m=\frac{\alpha_0}{3-2\alpha_0},\ n=\frac{\delta_0}{3-2\alpha_0},\ \text{respectively}\ \alpha_0=\frac{3m}{2m+1},\ \delta_0=\frac{3n}{2m+1}\ \text{at}\ m\neq-\frac{1}{2}$$

In the same way for the special case

$$\alpha(\tau)=\alpha_0, \quad \delta(\tau)=\delta_0, \tag{3.7}$$

we obtain

$$\omega(t)=\begin{cases}\omega_0\left(\dfrac{t}{t_0}\right)^n & n\neq\infty,\\ \omega_0\exp\left(\dfrac{1}{3}\delta_0\dfrac{t-t_0}{t_0}\right) & n=\infty,\end{cases} \tag{3.8}$$

and for $\Omega(t)$ is valid (5.10) in [1]. The case of boundary-layers defined by these forms of functions $\Omega(t)$ and $\omega(t)$ is solved in [3].

Now, let us still write the form of the function $\gamma(\tau)$ in mentioned cases. Namely, for (4.2) we obtain

$$\gamma(\tau)=\tau^s(\gamma_0+\gamma_1\tau+\gamma_2\tau^2+\cdots), \tag{3.9}$$

and for (3.8) and (5.10) from [1] it follows

$$\gamma(\tau)=\gamma_0\tau^s, \tag{3.10}$$

where

$$s=\frac{2n-m+1}{2m+1}.$$

As in this section, many problems of this type of boundary-layers, defined by other forms of functions $\Omega(t)$ and $\omega(t)$, can be considered.

4. Solutions of given systems of partial equations. In this section we shall give solutions of systems of partial equations (2.15) and (2.16). We shall consider only the case to which correspond components $\alpha(\tau)$ and $\delta(\tau)$ of a local principal function given by

$$\alpha(\tau)=\sum_{i=0}^{\infty}\alpha_i\tau^i, \quad \delta(\tau)=\sum_{k=0}^{\infty}\delta_k\tau^k. \tag{4.1}$$

In the paper [1] and in previous section we have seen that to them, in uniform correspondence following forms of the main-stream and angular velocity

$$\Omega(t)=t^m\sum_{i=0}^{\infty}\Omega_i t^{i(2m+1)}, \quad \omega(t)=t^n\sum_{k=0}^{\infty}\omega_k t^{k(2m+1)}, \tag{4.2}$$

belong, with $m, n\in[0,\infty)$, Ω_0, $\omega_0\neq0$, and otherwise all Ω_k and ω_k, where $(k=0,1,\ldots)$, are arbitrary.

Let us now give the required solutions. Observing the right side of (2.15) and (2.16), we see that only the forms of local principal functions $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$, determine the suitable forms of solutions $\mathfrak{F}_0(\eta, \tau), \ldots,$ $\mathfrak{R}_0(\eta, \tau), \ldots$ The forms of functions $\alpha(\tau)$ and $\beta(\tau)$ are given in [1] by (5.li) and by (5.11 i). But, the form of function $\gamma(\tau)$ is given by (3.9). The forms of functions $\alpha(\tau)$ and $\beta(\tau)$ will cover each other putting $k=0$ in the expression for $\beta(\tau)$. Thus, only the forms of functions $\beta(\tau)$ and $\gamma(\tau)$ determine the solutions of these equations, considering that in case of $\mathfrak{F}_0(\eta, \tau)$ and $\mathfrak{R}_0(\eta, \tau)$, we have $k=0$, The solutions of equations of systems (2.15) and (2.16) which on the right side have the functions $\beta(\tau)$ can be found as

(4.3)
$$\mathfrak{F}_\lambda(\eta, \tau) = \sum_{i=\sigma\varkappa}^{\infty} \mathfrak{F}_{\lambda i}(\eta)\, \tau^i,$$
$$\mathfrak{R}_\sigma(\eta, \tau) = \sum_{i=\sigma\varkappa}^{\infty} \mathfrak{R}_{\lambda i}(\eta)\, \tau^i,$$

where λ is the ordinal of the function $\mathfrak{F}_\lambda(\eta, \tau)$ in the series expansion (2.13), and σ of $\mathfrak{R}_\sigma(\eta, \tau)$ in (2.14) while $\varkappa$ is the initial power in the series expansion for the function $\beta(\tau)$. From (5.11i) in [1], we see that $\varkappa=k$. But, the forms of solutions in mentioned systems having $\gamma(\tau)$ on the right side are

(4.4)
$$\overset{*}{\mathfrak{F}}_{(\sigma+1)}(\eta, \tau) = \sum_{i=\zeta+\sigma\varkappa}^{\infty} \overset{*}{\mathfrak{F}}_{(\sigma+1)i}(\eta)\, \tau^i,$$

where ζ is an initial power in the series expansion for $\gamma(\tau)$. From (3.9) it is obviously that $\zeta=s$. In our case defined by (4.1) from [1] and by (3.1) we have $\varkappa=1$ and $\zeta=1$ and according to this also solutions as

(4.5a)
$$\mathfrak{F}_\lambda(\eta, \tau) = \sum_{i=\lambda}^{\infty} \mathfrak{F}_{\lambda i}(\eta)\, \tau^i,$$
$$\mathfrak{R}_\sigma(\eta, \tau) = \sum_{i=\sigma}^{\infty} \mathfrak{R}_{\sigma i}(\eta)\, \tau^i,$$

respectively

(4.5b)
$$\overset{*}{\mathfrak{F}}_{(\sigma+1)}(\eta, \tau) = \sum_{i=\sigma+1}^{\infty} \overset{*}{\mathfrak{F}}_{(\sigma+1)i}(\eta)\, \tau^i,$$

for every λ, $\sigma=(0,1,2,\ldots)$.

By substitution of such assumed solutions in systems (2.15) and (2.16) we obtain recursive systems of ordinary differential equations to determine coefficients of above series expansions. Linear partial operators L and K with valuable coefficients, given in section 2, will be now reduced to ordinary linear differential operators with constant coefficients

(4.6)
$$(L)_{\lambda k} = \left(\frac{d^3}{d\eta^3}\right)_{\lambda k} - \sum_{i=0}^{k-\lambda} \alpha_i \left[\frac{d}{d\eta} + \eta \frac{d^2}{d\eta^2}\right]_{\lambda(k-i)} + \frac{3}{2}\eta\left(\frac{d^2}{d\eta^2}\right)_{\lambda k} - 3k\left(\frac{d}{d\eta}\right)_{\lambda k},$$

and

(4.7)
$$(K)_{\lambda k} = \left(\frac{d^2}{d\eta^2}\right)_{\lambda k} - \sum_{i=0}^{k-\lambda}\left[\alpha_i \eta \frac{d}{d\eta} + \delta_i\right]_{\lambda(k-i)} + \frac{3}{2}\eta\left(\frac{d}{d\eta}\right)_{\lambda k} - 3k\,(1)_{\lambda k}.$$

The operator $(L)_{\lambda k}$ respectively $(K)_{\lambda k}$ will be applied to a function $\mathfrak{F}_{\lambda k}(\eta)$ according to

$$L(\mathfrak{F}_{\lambda k})=\mathfrak{F}_{\lambda k}'''-\sum_{i=0}^{k-\lambda}\alpha_i[\mathfrak{F}_{\lambda(k-i)}'+\eta\,\mathfrak{F}_{\lambda(k-i)}'']+\frac{3}{2}\eta\,\mathfrak{F}_{\lambda k}''-3\,k\,\mathfrak{F}_{\lambda k}'. \tag{4.8}$$

Thus, mentioned recursive systems of ordinary differential equations can be written as

$$L(\mathfrak{F}_{0k})=-\alpha_k,\quad (k=0,\,1,\,2,\ldots),$$

$$L\,\mathfrak{F}(_{1k})=-\beta_k-\sum_{i=1}^{k}\beta_i\left[\sum_{j=0}^{k-i}(-\mathfrak{F}_{0j}'\mathfrak{F}_{0(k-i-j)}'+\mathfrak{F}_{0j}\,\mathfrak{F}_{0(k-i-j)}'')\right],\ (k=1,2,\ldots),$$

(4.9)

$$L(\mathfrak{F}_{1ak})=-\sum_{i=1}^{k}\beta_i\sum_{j=0}^{k-i}\mathfrak{F}_{0j}\,\mathfrak{F}_{0(k-i-j)}'',\quad (k=1,2,\ldots),$$

$$L(\overset{*}{\mathfrak{F}}_{1k})=-\sum_{i=1}^{k}\gamma_i\sum_{j=0}^{k=i}\mathfrak{R}_{0j}\,\mathfrak{R}_{0(k-i-j)},\quad (k=1,2,\ldots),$$

and

$$K(\mathfrak{R}_{0k})=0,\quad (k=0,\,1,\,2,\ldots),$$

$$K(\mathfrak{R}_{1k})=-\sum_{i=1}^{k}\beta_i\sum_{j=0}^{k-i}\mathfrak{F}_{0j}\,\mathfrak{R}_{0(k-i-j)}',\quad (k=1,2,\ldots), \tag{4.10}$$

$$K(\mathfrak{R}_{1ak})=-\sum_{i=1}^{k}\beta_i\sum_{j=0}^{k-i}(\mathfrak{F}_{0j}\,\mathfrak{R}_{0(k-i-j)}'-2\,\mathfrak{F}_{0j}'\,\mathfrak{R}_{0(k-i-j)})\Bigg],\quad (k=1,2,\ldots).$$

The equations for $\mathfrak{F}_{ok}(\eta)$ and $\mathfrak{F}_{1k}(\eta)$ are the same as those ones in [1] formula (6.4), and $\mathfrak{R}_{0k}(\eta)$ and $\mathfrak{R}_{1k}(\eta)$ as those ones in [2], formula (5.8). The boundary conditions are

(4.11)
$$\mathfrak{F}_{00}(0)=\mathfrak{F}_{00}'(0)=0,\qquad \mathfrak{F}_{\lambda k}(0)=\mathfrak{F}_{\lambda k}'(0)=0,$$
$$\mathfrak{F}_{00}(\infty)=1,\qquad \mathfrak{F}_{\lambda k}(\infty)=0,\quad (\lambda,k=1,\,2,\ldots),$$

and

(4.12)
$$\mathfrak{R}_0(0)=1,\qquad \mathfrak{R}_{\sigma k}(0)=0,$$
$$\mathfrak{R}_0(\infty)=0,\qquad \mathfrak{R}_{\sigma k}(\infty)=0,\quad (\sigma,k=1,\,2,\ \ldots).$$

As it is seen in the paper [1] and in [2] all equations for $\mathfrak{F}_{0k}(\eta),\ldots,$ $\mathfrak{R}_{0k}(\eta),\ldots,$ when $i,\,k\geqslant 1$ can be done universally relating to parameters $\alpha_i,\,\delta_i$ and β_k,γ_k, but cannot to α_0 and δ_0. The reduction to universal functions is achieved by linear combinations of these parameters and of solutions independent of them. For $\mathfrak{F}_{0k}(\eta)$ and $\mathfrak{F}_{1k}(\eta)$ it can be found in [1], formula (6.6), and for $\mathfrak{F}_{1ak}(\eta)$ is the same as for $\mathfrak{F}_{1k}(\eta)$. But, the reduction of $\overset{*}{\mathfrak{F}}_{1k}(\eta)$ can be achieved by combinations of following parameters for single function $\overset{*}{\mathfrak{F}}_{1k}(\eta)$, namely for

$$\overset{*}{\mathfrak{F}}_{11}«\gamma_1»,$$

$$\overset{*}{\mathfrak{F}}_{12}«\alpha_1\gamma_1,\ \gamma_1\delta_1,\ \gamma_2»,$$

$$\overset{*}{\mathfrak{F}}_{13}«\alpha_1^2\gamma_1,\ \alpha_2\gamma_1,\ \alpha_1\gamma_2,\ \alpha_1\gamma_1\delta_1,\ \gamma_1\delta_1^2,\ \gamma_1\delta_2,\ \gamma_2\delta_1,\ \gamma_3», \tag{4.13}$$

$$\overset{*}{\mathfrak{F}}_{14} «\alpha_1^3 \gamma_1, \ \alpha_1^2 \gamma_2, \ \alpha_1 \alpha_2 \gamma_1, \ \alpha_1 \gamma_3, \ \alpha_2 \gamma_2, \ \alpha_3 \gamma_1, \ \alpha_1^2 \gamma_1 \delta_1, \ \alpha_1 \gamma_1 \delta_1^2,$$
$$\alpha_1 \gamma_1 \delta_2, \ \alpha_1 \gamma_2 \delta_1, \ \alpha_2 \gamma_1 \delta_1, \ \gamma_1 \delta_1^3, \ \gamma_1 \delta_1 \delta_2, \ \gamma_1 \delta_3, \ \gamma_2 \delta_1^2, \ \gamma_2 \delta_2, \ \gamma_3 \delta_1, \ \gamma_4»,$$

and corresponding universal functions. Denoting universal functions by. $\overset{*}{\varphi} \ldots (\eta)$, where points on the left side correspond to parameters δ_i, we can write for instance

$$\overset{*}{\mathfrak{F}}_{11} = \gamma_1 \overset{*}{\varphi}_1$$

$$\overset{*}{\mathfrak{F}}_{12} = \alpha_1 \gamma_1 \overset{*}{\varphi}_{11} + \gamma_1 \delta_1 \, {}_1\overset{*}{\varphi}_1 + \gamma_2 \overset{*}{\varphi}_2, \tag{4.13}$$

$$\overset{*}{\mathfrak{F}}_{13} = \alpha_1^2 \gamma_1 \overset{*}{\varphi}_{111} + \alpha_2 \gamma_1 \overset{*}{\varphi}_{21} + \alpha_1 \gamma_2 \overset{*}{\varphi}_{12} + \alpha_1 \gamma_1 \delta_1 \, {}_1\overset{*}{\varphi}_{11} +$$
$$+ \gamma_1 \delta_1^2 \, {}_{11}\overset{*}{\varphi}_1 + \gamma_1 \delta_2 \, {}_2\overset{*}{\varphi}_1 + \gamma_2 \delta_1 \, {}_1\overset{*}{\varphi}_2 + \gamma_3 \overset{*}{\varphi}_3 .$$

Linear combinations for $\mathfrak{R}_{0k}(\eta)$ and $\mathfrak{R}_{1k}(\eta)$ can be found in [2], formulae (5.16) — (5.17), but we shall write here by which parameters for single functions $\mathfrak{R}_{0k}(\eta)$ and $\mathfrak{R}_{1k}(\eta)$ is achieved the reduction. For $\mathfrak{R}_{0k}(\eta)$ by

$$\mathfrak{R}_{00}«1»,$$
$$\mathfrak{R}_{01}«\alpha_1, \ \delta_1», \tag{4.14}$$
$$\mathfrak{R}_{02}«\alpha_1^2, \ \alpha_2, \ \alpha_1 \delta_1, \ \delta_1^2, \ \delta_2»,$$
$$\mathfrak{R}_{03}«\alpha_1^3, \ \alpha_1 \alpha_2, \ \alpha_3, \ \alpha_1^2 \delta_1, \ \alpha_2 \delta_1, \ \alpha_1 \delta_1^2, \ \alpha_1 \delta_2, \ \delta_1^3, \ \delta_1 \delta_2, \ \delta_3»,$$

and for $\mathfrak{R}_{1k}(\eta)$ (and $\mathfrak{R}_{1ak}(\eta)$) by

$$\mathfrak{R}_{11}\,(\mathfrak{R}_{a11})\,«\beta_1»,$$
$$\mathfrak{R}_{12}\,(\mathfrak{R}_{a12})\,«\alpha_1 \beta_1, \beta_1 \delta_1, \beta_2», \tag{4.15}$$
$$\mathfrak{R}_{13}\,(\mathfrak{R}_{a13})\,«\alpha_1^2 \beta_1, \alpha_1 \beta_2, \alpha_2 \beta_1, \alpha_1 \beta_1 \delta_1, \beta_1 \delta_1^2, \beta_1 \delta_2, \beta_2 \delta_1, \beta_3».$$

Denoting corresponding universal functions for $\mathfrak{R}_{ok}(\eta)$ by $.r \ldots (\eta)$ and for $\mathfrak{R}_{1k}(\eta)$ by $.g \ldots (\eta)$, where again points on the left side correspond to parameters δ_i, we have for instance

$$\mathfrak{R}_{00} = R_0,$$
$$\mathfrak{R}_{01} = \alpha_1 r_1 + \delta_1 \, {}_1r, \tag{4.14'}$$
$$\mathfrak{R}_{02} = \alpha_1^2 r_{11} + \alpha_2 r_2 + \alpha_1 \delta_1 \, {}_1r_1 + \delta_1^2 \, {}_{11}r + \delta_2 \, {}_2r,$$

and

$$\mathfrak{R}_{11} = \beta_1 g_1,$$
$$\mathfrak{R}_{12} = \alpha_1 \beta_1 g_{11} + \beta_1 \delta_1 \, {}_1g_1 + \beta_2 \delta_2 . \tag{4.15'}$$

Thus, we shall obtain now the new system of differential equations which are independent of parameters $\alpha_i, \delta_i, \beta_i$ and γ_i for $i \geqslant 1$. Taking into account that operators L and K given by (4.7) and (4.8) will now be reduced to

(4.16) $$L_k = \frac{d^3}{d\eta^3} - \alpha_0 \left(\frac{d}{d\eta} + \eta \frac{d^2}{d\eta^2}\right) + \frac{3}{2}\eta \frac{d^2}{d\eta^2} - 3k\frac{d}{d\eta},$$

and

(4.17) $$K_k = \frac{d^2}{d\eta^2} + \left(\frac{3}{2} - \alpha_0\right)\eta \frac{d}{d\eta} - (3k + \delta_0),$$

we write the new systems of differential equations for universal functions $\varphi_a \dots (\eta)$ and $\overset{*}{\varphi} \dots (\eta)$ as

$$\begin{aligned}
k=1 \quad & L_1(\varphi_{a1}) = -F_0 F_0'' \\
k=2 \quad & L_2(\varphi_{a11}) = \varphi_1' + \eta\varphi_1'' - (F_0 f_1'' + f_1 F_0'') \\
& L_2(\varphi_{a2}) = -F_0 F_0'' \\
k=3 \quad & L_3(\varphi_{a111}) = \varphi_{11}' + \eta\varphi_{11}'' - (F_0 f_{11}'' + f_1 f_1'' + f_{11} F_0'') \\
& L_3(\varphi_{a12}) = \varphi_2' + \eta\varphi_2'' - (F_0 f_1'' + f_1 F_0'') \\
& L_3(\varphi_{a21}) = \varphi_1' + \eta\varphi_1' - (F_0 f_2'' + f_2 F_0'') \\
& L_3(\varphi_{a3}) = -F_0 F_0'',
\end{aligned}$$

(4.18)
$$\begin{aligned}
k=1 \quad & L_1(\overset{*}{\varphi}_1) = -R_0^2 \\
k=2 \quad & L_2(\overset{*}{\varphi}_{11}) = \overset{*}{\varphi}_1{}' + \eta\overset{*}{\varphi}_1{}'' - 2 r_1 R_0 \\
& L_2({}_1\overset{*}{\varphi}_1) = -2 R_0\, {}_1r \\
& L_2(\overset{*}{\varphi}_2) = -R_0^2 \\
k=3 \quad & L_3(\overset{*}{\varphi}_{a111}) = \overset{*}{\varphi}_{11}{}' + \eta\overset{*}{\varphi}_{11}{}'' - (2 R_0 r_{11} + r_1^2) \\
& L_3(\overset{*}{\varphi}_{21}) = \overset{*}{\varphi}_1{}' + \eta\overset{*}{\varphi}_1{}'' - 2 R_0 r_2 \\
& L_3(\overset{*}{\varphi}_{12}) = \overset{*}{\varphi}_2{}' + \eta\overset{*}{\varphi}_2{}'' - 2 R_0 r_1 \\
& L_3({}_1\overset{*}{\varphi}_{11}) = {}_1\overset{*}{\varphi}_1{}' + \eta\, {}_1\overset{*}{\varphi}_1{}'' - (2 R_0\, {}_1r_1 + r_1\, {}_1r) \\
& L_3({}_{11}\overset{*}{\varphi}_1) = -(2 R_0\, {}_{11}r + {}_1r^2) \\
& L_3({}_2\overset{*}{\varphi}_1) = -2 R_0\, {}_2r \\
& L_3({}_1\overset{*}{\varphi}_2) = -2 R_0\, {}_1r \\
& L_3(\overset{*}{\varphi}_3) = -R_0^2,
\end{aligned}$$

and afterward for $.r \dots (\eta)$, $.g \dots (\eta)$ and $.g_a \dots (\eta)$ as

$$\begin{aligned}
k=0 \quad & K(R_0) = 0 \\
k=1 \quad & K_1(r_1) = \eta R_0' \\
& K_1({}_1r) = R_0 \\
k=2 \quad & K_2(r_{11}) = \eta r_1' \\
& K_2(r_2) = \eta R_0' \\
& K_2({}_1r_1) = \eta\, {}_1r' + r_1 \\
& K_2({}_{11}r) = {}_1r \\
& K_2({}_2r) = R_0 \\
k=3 \quad & K_3(r_{111}) = \eta r_{11}' \\
& K_3(r_{12}) = \eta (r_1' + r_2')
\end{aligned}$$

$$K_3(r_3)=\eta R_0' \qquad K_3({}_2r_1)=\eta\, {}_2r'+r_1$$
$$K_3({}_1r_{11})=\eta\, {}_1r_1'+r_{11} \qquad K_3({}_{111}r)={}_{11}r$$
$$K_3({}_1r_2)=\eta\, {}_1r'+r_2 \qquad K_3({}_{12}r)={}_1r+{}_2r$$
$$K_3({}_{11}r_1)=\eta\, {}_{11}r'+{}_1r_1 \qquad K_3({}_3r)=R_0$$

$$k=1 \quad K_1(g_1)=-F_0 R_0'$$
$$k=2 \quad K_2(g_{11})=\eta g_1'-(f_1 R_0'+F_0 r_1')$$
$$K_2({}_1g_1)=g_1-F_0\, {}_1r'$$
$$K_2(g_2)=-F_0 R_0'$$

$$k=3 \quad K_3(g_{111})=\eta g_{11}'-(F_0 r_{11}'+f_1 r_1'+f_{11} R_0')$$
$$K_3(g_{12})=\eta g_2'-(F_0 r_1'+f_1 R_0')$$
$$K_3(g_{21})=\eta g_1'-(F r_2'+f_2 R_0')$$
$$K_3({}_1g_{11})=(\eta\, {}_1g_1'+g_{11})-(F_0\, {}_1r_1'+f_1\, {}_1r')$$
$$K_3({}_{11}g_1)={}_1g_1-F_0\, {}_{11}r'$$
$$K_3({}_2g_1)=g_1-F_0\, {}_2r'$$
$$K_3({}_1g_2)=g_2-F_0\, {}_1r'$$
$$K_3(g_3)=-F_0 R_0'$$

$$k=1 \quad K_1(g_{a1})=-(F_0 R_0'-2F_0' R_0)$$
$$k=2 \quad K_2(g_{a11})=\eta g_{a1}'-(f_1 R_0'-2f_1' R_0+F_0 r_1'-2F_0' r_1)$$
$$K_2({}_1g_{a1})=g_{a1}-(F_0\, {}_1r'-2F_0'\, {}_1r)$$
$$\text{(4.19)} \qquad K_2(g_{a2})=-(F_0 R_0'-2F_0' R_0)$$

$$k=3 \quad K_3(g_{a111})=\eta g_{a11}'-(F_0 r_{11}'-2F_0' r_{11}+f_1 r_1'-2f_1' r_1+f_{11} R_0'-2f_{11}' R_0)$$
$$K_3(g_{a12})=\eta g_{a2}'-(F_0 r_1'-2F_0' r_1+f_1 R_0'-2f_1' R_0)$$
$$K_3(g_{a21})=\eta g_{a1}'-(F_0 r_2'-2F_0' r_2+f_2 R_0'-2f_2' R_0)$$
$$K_3({}_1g_{a11})=(\eta\, {}_1g_{a1}'+g_{a11})-(F_0\, {}_1r_1'-2F_0'\, {}_1r_1+f_1\, {}_1r'-2f_1'\, {}_1r)$$
$$K_3({}_{11}g_{a1})={}_1g_{a1}-(F_0\, {}_{11}r'-2F_0'\, {}_{11}r)$$
$$K_3({}_2g_{a1})=g_{a1}-(F_0\, {}_2r'-2F_0'\, {}_2r)$$
$$K_3({}_1g_{a2})=g_{a2}-(F_0\, {}_1r'-2F_0'\, {}_1r)$$
$$K_3(g_{a3})=-(F_0 R_0'-2F_0' R_0)$$

The boundary conditions are

$$\text{(4.20)} \qquad \begin{aligned} &F_0(0)=F_0'(0)=0, \quad f\ldots(0)=f'\ldots(0)=0, \quad \varphi\ldots(0)=\varphi'\ldots(0)=0,\ldots. \\ &F_0(\infty)=1, \quad f\ldots(\infty)=0, \quad \varphi\ldots(\infty)=0,\ldots, \end{aligned}$$

$$\text{(4.21)} \qquad \begin{aligned} &R_0(0)=1, \quad r\ldots(0)=0, \quad g\ldots(0)=0,\ldots, \\ &R_0(\infty)=0, \quad r\ldots(\infty)=0, \quad g\ldots(\infty)=0,\ldots \end{aligned}$$

All above equations in the systems (4.18) and (4.19) are of the parabolic type and can be solved in the same way as it is shown in the paper [1]. They can be tabulated by numerical methods too.

Let us now consider the special case defined by

$$\Omega(t)=\Omega_0 t^m, \quad \omega(t)=\omega_0 t^n. \tag{4.22}$$

The principal functions in this case have the forms

$$\alpha(\tau)=\alpha_0, \quad \delta(\tau)=\delta_0, \quad \beta(\tau)=\beta\tau^k, \quad \gamma(\tau)=\gamma\tau^s, \tag{4.23}$$

where

$$\alpha_0=\frac{3m}{2m+1}, \quad \delta_0=\frac{3n}{2m+1}, \quad k=\frac{m+1}{2m+1}, \quad s=\frac{2r-m+1}{2m+1},$$

$$\beta=\frac{3}{2m+1}\Omega_0 A^k, \quad \gamma=\frac{3}{2m+1}\frac{\omega_0^2}{\Omega_0}A^s, \quad A=\frac{\nu(2m+1)}{\Omega_0^2}.$$

The variables τ and η are reduced to

$$\tau=\frac{1}{A}t^{2m+1}, \quad \eta=2\sqrt{\frac{2m+1}{3}}\,\bar{\eta}, \tag{4.24}$$

where $\bar{\eta}$ is the variable of simular solutions.

From (4.23) it follows that the solutions of equations of the systems (2.15) and (2.16) ought to be found in the forms

$$\mathfrak{F}_\lambda(\eta,\tau)=\mathfrak{F}_{\lambda k}(\eta)\,\tau^{\lambda k}, \quad \mathfrak{R}_\sigma(\eta,\tau)=\mathfrak{R}_{\sigma k}(\eta)\,\tau^{\sigma k}, \tag{4.25}$$

and

$$\overset{*}{\mathfrak{F}}_{\sigma+1}(\eta,\tau)=\overset{*}{\mathfrak{F}}_{(s+\sigma k)}(\eta)\,\tau^{s+\sigma k}, \tag{4.26}$$

where λ, $\sigma=0, 1, 2, \ldots$. By introducing linear combinations

$$\mathfrak{F}_{\lambda k}(\eta)=\bar{\beta}^\lambda\,\bar{\mathfrak{F}}_\lambda(\eta), \quad \mathfrak{R}_{\sigma k}(\eta)=\bar{\beta}^\sigma\,\bar{\mathfrak{R}}_\sigma(\eta), \tag{4.27}$$

and

$$\overset{*}{\mathfrak{F}}_{(s+\sigma k)}(\eta)=\bar{\gamma}\,\bar{\beta}^\sigma\,\overset{*}{\bar{\mathfrak{F}}}_{1+\sigma}(\eta), \tag{4.28}$$

where $\bar{\beta}=\Omega_0 A^k$ and $\bar{\gamma}=\frac{\omega_0^2}{\Omega_0}A^s$ the systems of partial equations (2.15) and (2.16) will be reduced to

$$\begin{aligned} &\bar{L}_0(\bar{\mathfrak{F}}_0')=-4m,\\ &\bar{L}_k(\bar{\mathfrak{F}}_1')=-4(1-\bar{\mathfrak{F}}_0'^2+\bar{\mathfrak{F}}_0\bar{\mathfrak{F}}_0''),\\ &\bar{L}_k(\bar{\mathfrak{F}}_{1a}')=-4\bar{\mathfrak{F}}_0\bar{\mathfrak{F}}_0'',\\ &\bar{L}_s(\overset{*}{\bar{\mathfrak{F}}}_1')=-4\bar{\mathfrak{R}}_0^2, \end{aligned} \tag{4.29}$$

respectively to

$$\overline{K}_0(\overline{\mathfrak{R}}_0)=0,$$

(4.30)
$$\overline{K}_k(\overline{\mathfrak{R}}_1)=4(2\overline{\mathfrak{F}}_0'\overline{\mathfrak{R}}_0-\overline{\mathfrak{F}}_0\overline{\mathfrak{R}}_0'),$$

$$\overline{K}_k(\overline{\mathfrak{R}}_{1a})=-4\overline{\mathfrak{F}}_0\overline{\mathfrak{R}}_0',$$

where $\overline{L}$ and $\overline{K}$ are obtained from (2.15′) and (2.16′) by transformation of η to $\overline{\eta}$, considering that the operator $\overline{L}$ is a second order operator and ought to be applied to the first derivative of a given function. Thus, these operators have the forms

(4.31)
$$\overline{L}_\varkappa=\frac{d^2}{d\overline{\eta}^2}+2\overline{\eta}\frac{d}{d\overline{\eta}}-4(2m+1)\cdot\varkappa-4m,$$

respectively

(4.32)
$$\overline{K}_\varkappa=\frac{d^2}{d\overline{\eta}^2}+2\overline{\eta}\frac{d}{d\overline{\eta}}-4(2m+1)-4n,$$

and otherwise $\varkappa$ is either λk or $s+\sigma k$, in dependence to which function of (4.27) and (4.28) is related. For instance, in case $\lambda=1$ we have $\varkappa=k=\frac{m+1}{2m+1}$, and according to this

$$\overline{L}_k(\overline{\mathfrak{F}}_1')=\overline{\mathfrak{F}}_1'''+2\overline{\eta}\overline{\mathfrak{F}}_1''-4(2m+1)\overline{\mathfrak{F}}_1'.$$

The boundary conditions are

(4.33)
$$\overline{\mathfrak{F}}_0(0)=\overline{\mathfrak{F}}_0'(0)=0,\quad \overline{\mathfrak{F}}_\lambda(0)=\overline{\mathfrak{F}}_\lambda'(0)=\overline{\overset{*}{\mathfrak{F}}}_\lambda(0)=\overline{\overset{*}{\mathfrak{F}}}{}_\lambda'(0)=0,$$

$$\overline{\mathfrak{F}}_0'(\infty)=1,\quad \overline{\mathfrak{F}}_\lambda'(\infty)=\overline{\overset{*}{\mathfrak{F}}}{}_\lambda'(\infty)=0,\quad (\lambda=1,2,\ldots),$$

(4.34)
$$\overline{\mathfrak{R}}_0(0)=1,\quad \overline{\mathfrak{R}}_\sigma(0)=0,$$

$$\overline{\mathfrak{R}}_0(\infty)=0,\quad \overline{\mathfrak{R}}_\sigma(\infty)=0,\quad (\sigma=1,2,\ldots).$$

In this case the expressions for functions $\mathfrak{F}(x,\eta,\tau)$ and $\mathfrak{R}(x,\eta,\tau)$ being

(4.35)
$$\mathfrak{F}(x,\overline{\eta},\tau)=\overline{\mathfrak{F}}_0(\overline{\eta})+\Omega_0 t^{m+1}\left[V'\overline{\mathfrak{F}}_1(\overline{\eta})+V\frac{r'}{r}\overline{\mathfrak{F}}_{1a}(\overline{\eta})\right]+$$

$$+\frac{\omega_0^2}{\Omega_0}\frac{rr'}{V}t^{2n-m+1}\overline{\overset{*}{\mathfrak{F}}}_1(\overline{\eta})+\cdots,$$

(4.36)
$$\mathfrak{R}(x,\overline{\eta},\tau)=\overline{\mathfrak{R}}_0(\overline{\eta})+\Omega_0 t^{m+1}\left[V'\overline{\mathfrak{R}}_1(\overline{\eta})+V\frac{r'}{r}\overline{\mathfrak{R}}_{1a}(\overline{\eta})\right]+\cdots$$

The equations (4.29) and (4.30) as well as the form of functions $\mathfrak{F}(x,\eta,\tau)$ and $\mathfrak{R}(x,\eta,\tau)$ are quite the same as those obtained in the paper [3] at solving the same problem. The solutions of these equations for the boundary conditions (4.33) and (4.34) can be found in [3].

5. The time interval before separation begins and the distance which the body covers before separation commences. The sink friction has components

(5.1) $$N_x = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad N_z = \mu \left(\frac{\partial w}{\partial y}\right)_{y=0},$$

in x respectively z direction.

The position of separation at a given time can be calculated once when the external velocity distribution is specified, and occurs when

(5.2) $$N_x = 0,$$

namely

(5.3) $$\mathfrak{F}_{\eta\eta}(x, 0, \tau) = 0,$$

and $\mathfrak{F}(x, \eta, \tau)$ is given by (2.13).

In the general case it is more convenient to consider the distance s which the body covers before separation commences, rather then the time t_s. This distance we find from the expression

(5.4) $$s = \int_0^t \Omega(t)\, dt,$$

when instead of t we put t_s the time of separation begins, and this time is to be found as the solution of the equation (5.3).

6. Conclusion. — In conclusion, we shall say still a few words about the method. About the convergence we can speak only principally. In the same way as in [1] we see that the convergence of series expansions (2.13) and (2.14), independently of coefficients $\mathfrak{F}_\lambda(\eta, \tau)$, $\mathfrak{R}_\sigma(\eta, \tau)$ and $\overset{*}{\mathfrak{F}}_{\sigma+1}(\eta, \tau)$, where $(\lambda, \sigma = 0, 1, \ldots)$, will be better when the angle of attack of the body in the forward stagnation point is nearer to zero. But, on the convergence τ has influence too. In [1] we have seen that τ must be < 1. We cannot say anything about the extension of the domain of outside 1, on account of reasons quoted in [1]. To the purpose of application of the method one ought to make the tabulation of universal functions. Then, the way for its application is the same as in [1]. But, one can put the remark about the method on account of too great number of universal functions, which do that the method, although for application very simple, is for numerical calculation rather a long one. But, in the case of axi-symmetrical boundary-layers from equations (2.9) and (2.10) remains only the first one without the part $\gamma(\tau)\frac{rr'}{V}\mathfrak{R}^2$. In the set of principal functions remain only $\alpha(\tau)$ and $\beta(\tau)$. The series expansions (2.13) and (2.14) are reduced to

$$\mathfrak{F}(x, \eta, \tau) = \mathfrak{F}_0(\eta, \tau) + V'\, \mathfrak{F}_1(\eta, \tau) + V\frac{r'}{r}\mathfrak{F}_{1a}(\eta, \tau) + \cdots,$$

$$\mathfrak{R}(x, \eta, \tau) = 0.$$

That means that relating to the method [1] we have an extension in the part $V\frac{r'}{r}\mathfrak{F}_{1a}$ and equivalent parts which follow further in the series expansion for $\mathfrak{F}(x, \eta, \tau)$. From equations (2.15) and (2.16) remain only the first three ones of (2.15) i. e. for $\mathfrak{F}_0(\eta, \tau)$, $\mathfrak{F}_1(\eta, \tau)$ and $\mathfrak{F}_{1a}(\eta, \tau)$. Thus, the number of universal functions will be lesser and the method more simplier for numerical calculation. We were not in possibility to make the tabulation of universal functions on account of technical reasons.

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