PUBLICATIONS L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 7 (21) 1967 163–168

ONE PARAMETER SOLUTION OF THERMAL BOUNDARY–LAYERS PAST A FLAT PLATE

Milan Đ. Đurić

(Communicated Sept. 28, 1966)

Abstract. This paper presents a solution of thermal boundary-layer equation, when it is assumed that the main-stream velocity is defined by the function $\Omega(t)$, and the difference between the wall and main-stream temperature by $\theta(t)$. The velocity distribution inside the boundary-layer is taken as an exact one, from the class of exact solutions, for instance, from [2], while the temperature distribution is obtained as the approximate one, depending on a parameter — the temperature local shape parameter.

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1. Introduction. — In the paper [3] is given the general method for solution of thermal boundary-layers in case of two-dimensional low-speed flows. But, in comparison to that method, in this paper, starting from momentum integral equation is given a one-parameter solution for a flat plate. Namely, taking the velocity distribution inside the boundary-layer as an exact one, we have obtained the temperature distribution as the approximate one, depending on the temperature shape parameter. The main-stream velocity is assumed in the form $\Omega(t)$ and the difference between the temperature of the wall and that of the main-stream as $T_w(t)-T_\infty=\theta(t)$. Parallel are considered the heating, respectively cooling, and thermometer problem. The family of temperature profiles is given by the quartic polinomials, chosen to satisfy certain boundary conditions. Obviously, this solution is more complicated than the corresponding one in the paper [3].

2. The momentum integral equation and its transformation. — The thermal equation for forced-convection flows past a flat plate is

$$\frac{\partial T}{\partial t}=\frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{gc_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{2.1}$$

where y is the distance normal to the wall; $u\ (y, t)$ is the velocity component in direction of the wall; $T(y, t)$ is the temperature field; t the time; ν the coefficient of kinematic viscosity; c_p the specific heat at constant pressure and σ the Prandtl number.

The boundary conditions being

$$\begin{aligned} &T=T_w(t) \text{ or } \frac{\partial T}{\partial y}=0, && y=0,\\ &T=T_\infty, && y=\infty. \end{aligned} \tag{2.2}$$

Integrating the equation (2.1) across the thermal-layer from $y=0$ to $y=\infty$, respectively to $y=\delta_T$, where δ_T is the thermal-layer thickness, and introducing a measure of the thermal-layer

$$\delta_T^* = \int_0^{\infty,\, \delta_T} \left(1-\frac{T}{T_\infty}\right) dy, \tag{2.3}$$

at which we consider that the above integral exists, and the local heat-transfer rate

$$q = -\lambda \left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{2.4}$$

we obtain as a result the momentum integral equation

$$\frac{d}{dt}(\delta_T^* T_\infty) = -\frac{\nu}{\sigma}\frac{1}{\lambda} q - \frac{\nu}{g c_p} \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy. \tag{2.5}$$

To solve this equation we ought to know the velocity field. The method [2] is available for this purpose. Namely, from [2], and in case of an unsteady velocity boundary-layer flows past a flat plate, for the velocity distribution inside the boundary layer we have the following expression

$$u(y,\ t) = \Omega(t)\, \mathfrak{F}_{o\eta}(\eta,\ \tau). \tag{2.6}$$

where η and $\tau=\tau(t)$ are given in [2] by (2.6). Moreover, for solving the equation (2.5) we ought to express the unknown magnitudes δ_T^* and q only over a magnitude f_T. To this purpose we assume the temperature distribution inside the thermal boundary layer as

$$T(y,\ t) = T_\infty + (T_w(t) - T_\infty)\, \mathcal{H}(\eta_T;\ f_T) + \frac{1}{g c_p}\, \Omega^2(t)\, \mathcal{K}(\eta_T;\ h_T). \tag{2.7}$$

where $\eta_T = \dfrac{y}{\delta_T^*}$, $h_T = h_T(f_T)$ and f_T is a parameter which is to be determined from a condition

$$f_T = \left(\frac{d^2 \mathcal{H}}{d\eta_T^2}\right)_{\eta_T=0} = \frac{\delta_T^{*2}\dot{\theta}}{\nu\theta}, \tag{2.8}$$

considering that the difference between the temperature of the wall and that of the main-stream is given by a function $\theta(t)$, namely $T_w(t) - T_\infty = \theta(t)$. Now, we have

$$q = -\lambda \frac{\theta}{\delta_T^*} \zeta_T(f_T), \text{ where } \zeta_T(f_T) = \mathcal{H}(0;\ f_T). \tag{2.9}$$

On account of (2.8) and (2.9) we reduce the equation (2.5) to the new form

$$f_T' + f_T\left(\frac{\dot{\theta}}{\theta} - \frac{\ddot{\theta}}{\dot{\theta}}\right) = \frac{2}{\sigma T_\infty}\dot{\theta}\zeta_T - \frac{2}{g c_p}\frac{1}{T_\infty}\sqrt{\frac{\dot{\theta}}{\theta}}\frac{\Omega^3}{\left(3\int_0^t \Omega^2\, dt\right)^{1/2}} \sqrt{f_T} \int_0^\infty \mathfrak{F}_{o\eta\eta}^2\, d\eta. \tag{2.10}$$

It remains still, to find the functional relation between the local heat-transfer rate q and the parameter f_T, namely the function $\zeta_T=\zeta_T(f_T)$.

3. The temperature distribution and some magnitudes of the thermal boundary layer. To obtain a procedure for solving the equation (2.10) we assume a temperature distribution inside the boundary-layer of the form

$$T-T_\infty=(T_w-T_\infty)\mathcal{H}(\eta_T;\ f_T)+\frac{1}{gc_p}\,\Omega^2\,\mathcal{K}(\eta_T;\ h_T), \tag{3.1}$$

where $\mathcal{K}(\eta_T;\ f_T)$ and $\mathcal{K}(\eta_T;\ h_T)$ are assumed as the quartic polynomials

$$\mathcal{H}(\eta_T;\ f_T)=a_0+a_1\,\eta_T+a_2\,\eta_T^2+a_3\,\eta_T^3+a_4\,\eta_T^4, \tag{3.2}$$

and

$$\mathcal{K}(\eta_T;\ h_T)=b_0+b_1\,\eta_T+b_2\,\eta_T^2+b_3\,\eta_T^3+b_4\,\eta_T^4. \tag{3.3}$$

On purpose of determining unknown coefficients in above polynomials, we add to the boundary conditions (2.2) still the following ones

$$\begin{gathered}\left(\frac{\partial T}{\partial t}\right)_{y=0}=\frac{\nu}{\sigma}\left(\frac{\partial^2 T}{\partial y^2}\right)_{y=0}+\frac{\nu}{gc_p}\left(\frac{\partial u}{\partial y}\right)^2_{y=0},\\ \left(\frac{\partial T}{\partial y}\right)_{y=\infty}=\left(\frac{\partial^2 T}{\partial y^2}\right)_{y=\infty}=0.\end{gathered} \tag{3.4}$$

We choose the boundary conditions so that $\mathcal{H}$ represents the solution of heating, respectively cooling, and $\mathcal{K}$ of the thermometer problem. Thus, on account of (3.4) and (2.2) we conclude that these functions ought to satisfy the following conditions

$$\begin{aligned}&\mathcal{H}=1,\quad \dot{\theta}\mathcal{H}+\theta\mathcal{H}_t=\frac{\nu}{\sigma}\theta\frac{1}{\delta_T^{*2}}\mathcal{H}_{\eta_T\eta_T}, && \eta_T=0,\\ &\mathcal{H}=0,\quad \mathcal{H}_{\eta_T}=\mathcal{H}_{\eta_T\eta_T}=0, && \eta_T=1,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\mathcal{K}_\eta=0,\quad 2\,\Omega\dot{\Omega}\,\mathcal{K}+\Omega^2\,\mathcal{K}_t=\frac{\nu}{\sigma}\,\Omega^2\,\frac{1}{\delta_T^{*2}}\,\mathcal{K}_{\eta_T\eta_T}+\Omega^4\left(3\int_0^t\Omega^2\,dt\right)^{-\frac{1}{2}}\mathfrak{F}^2_{0\eta\eta}, && \eta_T,\eta=0\\ &\mathcal{K}=0,\quad \mathcal{K}_{\eta_T}=\mathcal{K}_{\eta_T\eta_T}=0, && \eta_T=\infty.\end{aligned} \tag{3.6}$$

Satisfying above conditions we see that the functions (3.2) and (3.3) take the forms

$$\mathcal{H}(\eta_T;\ f_T)=(1-2\,\eta_T+2\,\eta_T^3-\eta_T^4)+f_T\frac{\sigma}{2}\left(-\frac{1}{3}\,\eta_T+\eta_T^2-\eta_T^3+\frac{1}{3}\,\eta_T^4\right), \tag{3.7}$$

respectively

$$\mathcal{K}(\eta_T;\ h_T)=h_T(1-6\,\eta_T^2+8\,\eta_T^3-3\,\eta_T^4), \tag{3.8}$$

where the function

$$h_T = b_0 = \frac{1}{3\nu} \frac{1}{\Omega^2} \exp\left(-12 \frac{1}{\sigma} \int \frac{1}{f_T} \frac{\dot{\theta}}{\theta}\, dt\right) \tag{3.9}$$

$$\int_0^t \frac{\Omega^4}{\int_0^t \Omega^2\, dt} \mathfrak{F}^2_{0\eta\eta}(0,\ \tau) \exp\left(12 \frac{1}{\sigma} \int \frac{1}{f_T} \frac{\dot{\theta}}{\theta}\, dt\right) dt,$$

is obtained as a solution of the following differential equation

$$h'_T + h_T \left(2 \frac{\dot{\Omega}}{\Omega} + 12 \frac{\nu}{\sigma} \frac{1}{\delta_T^{*2}}\right) - \frac{\Omega^2}{3 \int_0^t \Omega^2\, dt} \mathfrak{F}^2_{o\eta\eta}(0,\ \tau) = 0. \tag{3.10}$$

This differential equation is obtained from (3.3) and (3.6) taking into account that $b_2 = -6\, b_0$.

Now, we are able to give the required functional relation $\zeta_T = \zeta_T(f_T)$, and according to that, also, the local heat-transfer rate

$$q = -\lambda \frac{\theta}{\delta_T^*} \left(2 + \frac{1}{6}\, \sigma f_T\right). \tag{3.11}$$

But, to obtain a general temperature distribution which will satisfy all stated conditions, we ought to take into account that the frictional heating produces the increase of the wall temperature to a certain value $\hat{T}_w(t)$, which accordingly to (3.1) is determined by

$$\hat{T}_w(t) - T_\infty = \frac{1}{g c_p} \Omega^2(t)\, \mathcal{K}(0;\ h_T). \tag{3.12}$$

This temperature is the proper temperature of the wall, which because of (3.8) has the following value

$$\hat{T}_w - T_\infty = \frac{1}{g c_p} \Omega^2 h_T. \tag{3.13}$$

Thus, the general solution is to be taken as

$$T(y,\ t) - T_\infty = [(T_w(t) - T_\infty) - (\hat{T}_w(t) - T_\infty)]\, \mathcal{H}(\eta_T;\ f_T) + \tag{3.14}$$

$$+ \frac{1}{g C_p} \Omega^2(t)\, \mathcal{K}(\eta_T;\ h_T),$$

respectively, when is divided by $T_w - T$, it is

$$\frac{T - T_\infty}{T_w - T_\infty} = (1 - \Theta\, h_T)\, \mathcal{H} + \Theta\, \mathcal{K}, \tag{3.15}$$

where

$$\Theta(t) = \frac{1}{g\,c_p}\,\frac{\Omega^2(t)}{\theta(t)},$$

is the temperature criterion. On the base of Θ we can determine the direction of heat-transfer from the temperature gradient. So, if the product of $\Theta h_T < 1$, then we have that $T_w - T_\infty > 0$, namely, the temperature of the wall is higher than that one of the main-stream, what means that the heat-transfer direction is from the wall to the fluid. Thus, the fluid flow past the flat plate produces its cooling.

Now, we can determine as well as in [3], also here, some characteristic magnitudes of the thermal-layer. So, for instance, the heat-transfer rate from the flowed surface-area A of the body to the fluid, in any time t, taking the frictional heating in consideration, is

$$\hat{q}(t) = -\lambda\,\frac{T_w(t) - \hat{T}_w(t)}{\delta_T^*}\,\mathcal{H}_{\eta_T}(0;\, f_T), \tag{3.16}$$

while other magnitudes we will not write, although they can be determined very easily.

4. Solving the reduced momentum integral equation. — We have made all preparation for solving the equation (2.10). Taking into account (3.11) and introducing a new parameter Λ instead of f_T by $\Lambda^2 = f_T$, we reduce the mentioned equation to the form

$$\Lambda\Lambda' = f_2\,\Lambda^2 + f_1\,\Lambda + f_0, \tag{4.1}$$

where

$$f_2(t) = \frac{1}{2}\left(\frac{1}{3}\,\frac{1}{T_\infty}\,\dot{\theta} + \frac{\ddot{\theta}}{\dot{\theta}} - \frac{\dot{\theta}}{\theta}\right),$$

$$f_1(t) = -\frac{1}{g c_p T_\infty}\left(\frac{\dot{\theta}}{\theta}\right)^{1/2}\Omega^3\left(3\int_0^t \Omega^2\,dt\right)^{-1/2}\int_0^\infty \mathfrak{F}_{o\eta\eta}^2\,d\eta,$$

$$f_0(t) = \frac{2}{\sigma T_\infty}\,\dot{\theta}.$$

To obtain a more suitable form for solving the above equation it is necessary to make its reduction. Namely, by a substitution

$$\Lambda = E^{-1}(\overset{*}{\Lambda} + F), \tag{4.2}$$

where

$$E = \exp\left(-\int f_2\,dt\right) = \left(\frac{\theta}{\dot{\theta}}\right)^{1/2}\exp\left(-\frac{1}{6\,T_\infty}\,\theta\right),$$

$$F = \int f_1\,E\,dt = -\frac{1}{g c_p T_\infty}\int \Omega^3\left(3\int_0^t \Omega^2\,dt\right)^{-1/2}\left(\int_0^\infty \mathfrak{F}_{o\eta\eta}^2\,d\eta\right)\exp\left(-\frac{1}{6\,T_\infty}\,\theta\right)dt,$$

we obtain a new differential equation

$$\overset{*}{\Lambda}{}'\,(\overset{*}{\Lambda} + F) = H, \tag{4.3}$$

at which

$$H = f_0 E^2 = \frac{2}{\sigma T_\infty} \theta \exp\left(-\frac{1}{3 T_\infty} \theta\right).$$

Moreover, if we put

$$\overset{*}{\Lambda} = A(\xi), \quad \xi = \int H \, dt, \tag{4.4}$$

then the above equation will be reduced to the form

$$A'(A + F) = 1. \tag{4.5}$$

This equation ought to be solved from case to case, because there is no solution of it in the closed form. But, when a solution of it is found, then we can find the required thermal shape parameter

$$f_T = E^{-2}(A + F)^2, \tag{4.6}$$

and accordingly that also the temperature distribution and other magnitudes of the thermal layer.

5. Conclusion. — Now, we shall observe the obtained temperature distribution. On account of (3.7) — (3.9) and (4.6) we conclude that the solution (3.15) is more complicated than corresponding one in the paper [3]. Thus, one-parameter methods, in comparison to those ones in case of velocity boundary-layer flows, are less suitable for solving the problems of thermal boundary-layer flows, than the methods from the class of exact solutions, as, it is the method [3]. But, it is possible to make the one-parameter solution more simpler taking less complicated forms of the temperature and velocity distribution inside boundary layers. But, we will not occupy ourselves with it in this paper.

REFERENCES

[1] Curle, N., *The laminar boundary layer equations.* Oxford Math. Monogr., (1962).

[2] Đurić, M., *A method for solution of unsteady incompressible laminar boundary layers.* Publ. Inst. Math. (Beograd), 6 (20), (1966), 29-55.

[3] Đurić, M., *On the method for solution of unsteady thermal boundary layers in case of two-dimensional low-speed flows.* (to appear).