PUBLICATIONS DE L'INSITUT MATHÉMATIQUE
Nouvelle série, tome 7 (21), 1967. p. 139—161

LAMINAR BOUNDARY LAYER ON CYLINDRICAL BODIES STARTED FROM CERTAIN PRECEDING NON-STEADY MOTIONS

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(Communicated June 29, 1966)

1. Introduction. — Soon after the basic equations of the boundary layer theory were laid down, and concurrently with problems of plane steady motion, attempts were made to solve the simplest problems of a non-steady boundary layer. Nevertheless, we can say that the steady boundary layers have been investigated in far greater detail than the non-steady ones. The first studies of the non-steady boundary layer were made immediately after the creation of Prandtl's boundary layer theory by Blasius (1), one of Prandtl's co-workers; in that paper, Blasius studied the boundary layer on a cylindrical body which was started to a motion through a still and viscous fluid by an impulse. Blasius also solved the problem of development of a boundary layer on a cylinder brought to a motion by constant acceleration when starting from rest. Goldstein and Rosenhead (2) calculated several successive approximations and thus completed Blasius' solutions. Görtler (3) provided a solution of the non-steady boundary layer on a cylindrical body with the increase of velocity with time given by power series. Watson (4) gave a more generalized form of Görtler's solution for a case of an arbitrary exponent in power series accelerated motion of a cylinder, and also for a case of motion given by an exponential law for the change of rate of velocity with time. M. Djurić (5) introduced a method for solving the problem of a non-steady laminar boundary layer when the external velocity is given by a form of $U(x, t) = V(t) W(x)$.

The general and common feature of all the existing papers on non-steady boundary layers is — the state of rest prior to motion of a cylindrical body. The body and the fluid remain at rest until a certain moment, and then either the body starts to move through the still and viscous fluid, or the fluid begins to flow past the body at rest.

Problems of a boundary layer on a body in motion started not from rest but ensuing from a certain preceding motion have not been dealt with, and in literature there is no trace of any attempt to solve such problems.

The subject of the present paper is the boundary layer on a cylindrical body brought into motion from a state of certain non-steady motion (in further discussion this motion will be referred to as preceding motion). Therefore, a body was in motion, starting from the state of rest by an impulse and being either constantly accelerated or power series accelerated. Because of this „preceding" motion, a preceding non-steady boundary layer developed on the body. In the course of such a motion, at a certain instant $t=T$, this initial motion is given an „additional" accelerated motion which will bring about a certain state of velocities in the boundary layer, which we now propose to investigate.

Obviously, the problem can be set up in two different ways. Either to seek a solution of well — known partial equations of the non-steady boundary layer following the instant of $t=T$, and to require that such solutions, in addition to fulfilling the boundary conditions, fulfil also the inital one, that for $t=T$ they be reduced to the well-known solutions of the preceding boundary layer, or to find out only those „additional" velocity which determine the „additional" boundary layer — through which the existing boundary layer on the body at the instant $t=T$ was „completed" due to the additional motion.

The first of these approaches is ruled out from the very beginning since it involves practically insurmountable difficulties. Firstly, because for the solution of the non-steady boundary layer equations in terms of total velocities after an instant $t=T$, it is not possible to use the only known method of solution by successive approximations. Secondly, because even if we could find the general solutions of these equations, the fulfilment of the initial condition mentioned above would be practically unfeasible due to the extreme complexity of the solution of the preceding boundary layer.

Hence the second approach is more acceptable because, first of all, the initial conditions for the additionary boundary layer would be universally defined, and for the solution of equations for the additional boundary layer a method of successive approximations could be set up.

We proceed now to find equations for the additional boundary layer. Here we shall use the idea that the preceding boundary layer formally continued its development even after the instant $t=T$ within the total velocities in the boundary layer.

2. Derivation of equations of the additional boundary layer. — We start from the fundamental Navier-Stokes equations of hydromechanics for a two--dimensional flow:

$$
(2.1) \qquad \left\{\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta v\\
&\qquad\qquad\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0
\end{aligned}\right.
$$

and we transform them into a non-dimensional form by using a characteristic length l and a characteristic velocity V:

$$
(2.2) \qquad \begin{aligned}
&x=l\bar{x}, \quad y=l\bar{y}, \quad t=\frac{l}{V}\bar{t},\\
&u=V\bar{u}, \quad v=V\bar{v}, \quad p=\rho V^2\bar{p}, \quad \nu=\frac{lV}{R_e}
\end{aligned}
$$

By substituting (2.2) in (2.1) and dividing both sides of the first two equations of the system by $\frac{V^2}{l}$, and both sides of the third equation by $\frac{V}{l}$, we obtain

$$
(2.3) \quad \begin{cases} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{R_e}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \end{cases}
$$

where the dashes above symbols for physical quantities and coordinates have been omitted. Thus, we obtain a parameter $R_e=\frac{Vl}{\nu}$ — the Reynolds number.

If, similarly to Mises' method for the derivation of Prandtl's (6) equations, we now express (2.3) in a curvilinear orthogonal coordinate system (q_1, q_2) we have

$$
\frac{\partial v_1}{\partial t}+\frac{v_1}{H_1}\frac{\partial v_1}{\partial q_1}+\frac{v_2}{H_2}\frac{\partial v_1}{\partial q_2}+\frac{v_2}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial q_2}-v_2\frac{\partial H_2}{\partial q_1}\right)=
$$

$$
=-\frac{1}{H_1}\frac{\partial p}{\partial q_1}+\frac{1}{R_e}\left[\frac{1}{H_1^2}\frac{\partial^2 v_1}{\partial q_1^2}+\frac{1}{H_2^2}\frac{\partial^2 v_1}{\partial q_2^2}+\frac{1}{H_1H_2}\frac{\partial(H_2/H_1)}{\partial q_1}\frac{\partial v_1}{\partial q_1}+\right.
$$

$$
+\frac{1}{H_1H_2}\frac{\partial(H_1/H_2)}{\partial q_2}\frac{\partial v_1}{\partial q_2}+\frac{2}{H_1^2H_2}\frac{\partial H_1}{\partial q_2}\frac{\partial v_2}{\partial q_1}-\frac{2}{H_1H_2^2}\frac{\partial H_2}{\partial q_1}\frac{\partial v_2}{\partial q_2}+\frac{1}{H_1}\frac{\partial}{\partial q_1}\left(\frac{1}{H_1H_2}\frac{vH_2}{\partial q_1}\right)v_1+
$$

$$
\left.+\frac{1}{H_2}\frac{\partial}{\partial q_2}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial q_2}\right)v_1+\frac{1}{H_1}\frac{\partial}{\partial q_1}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial q_2}\right)v_2-\frac{1}{H_2}\frac{\partial}{\partial q_2}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial q_1}\right)v_2\right]
$$

(2.4)

$$
\frac{\partial v_2}{\partial t}+\frac{v_1}{H_1}\frac{\partial v_2}{\partial q_1}+\frac{v_2}{H_2}\frac{\partial v_2}{\partial q_2}-\frac{v_1}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial q_2}-v_2\frac{\partial H_2}{\partial q_1}\right)=-\frac{1}{H_2}\frac{\partial p}{\partial q_2}+
$$

$$
+\frac{1}{R_e}\left[\frac{1}{H_1^2}\frac{\partial^2 v_2}{\partial q_1^2}+\frac{1}{H_2^2}\frac{\partial^2 v_2}{\partial q_2^2}+\frac{1}{H_1H_2}\frac{\partial(H_2/H_1)}{\partial q_1}\frac{\partial v_2}{\partial q_1}+\frac{1}{H_1H_2}\frac{\partial(H_1/H_2)}{\partial q_2}\frac{\partial v_2}{\partial q_2}-\right.
$$

$$
-\frac{2}{H_1^2H_2}\frac{\partial H_1}{\partial q_2}\frac{\partial v_1}{\partial q_1}+\frac{2}{H_1H_2^2}\frac{\partial H_2}{\partial q_1}\frac{\partial v_1}{\partial q_2}+\frac{1}{H_1}\frac{\partial}{\partial q_1}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial q_1}\right)v_2+
$$

$$
\left.+\frac{1}{H_2}\frac{\partial}{\partial q_2}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial q_2}\right)v_2-\frac{1}{H_1}\frac{\partial}{\partial q_1}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial q_2}\right)v_1+\frac{1}{H_2}\frac{\partial}{\partial q_2}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial q_1}\right)v_1\right]
$$

$$
H_2\frac{\partial v_1}{\partial q_1}+H_1\frac{\partial v_2}{\partial q_2}+v_1\frac{\partial H_2}{\partial q_1}+v_2\frac{\partial H_1}{\partial q_2}=0.
$$

where H_1 and H_2 are the corresponding Lamé coefficients. Let us select a system of curvilinear coordinates according to Fig. 1. In points of the contour

C we draw normals to that contour, and let the normal through an arbitrary point M, which is located near the contour C, meet the contour at the point N. If we select on the contour a fixed point O (e. g. the front stagnation point of the body's contour), the position of the arbitrary point M with respect to that fixed point is determined by coordinates $q_1 = s$, $q_2 = n$, where s is the length of the arc ON, while n is the length of the normal NM. Find the distance between the adjacent points M and M'. Infinitely close normals MN and $M'N'$ meet at the center of curvature K of the curve C, corresponding to the point N. Denote by $r(s)$ the radius of curvature of C at the point N, and assume that $r(s)$ is of a continuous function of the variable s, complete with its first derivative. Then, by taking the square of the elementary distance MM', we obtain the following values of Lamé's coefficients:

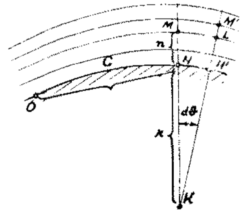

Fig. 1.

(2.5)
$$H_1 = 1 + \frac{n}{r(s)}, \quad H_2 = 1.$$

If we now introduce the relations

(2.6)
$$\begin{cases} q_1 = s = x, \quad q_2 = n = \dfrac{y}{\sqrt{R_e}}; \quad v_1 = v_s = u, \quad v_2 = v_n = \dfrac{v}{\sqrt{R_e}}; \\ H_1 = 1 + \dfrac{y}{\sqrt{R_e}\, r(x)}, \quad H_2 = 1; \end{cases}$$

and if we substitute (2.6) in (2.4), we obtain

(2.7)
$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{H_1} u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} + \dfrac{1}{r\sqrt{R_e}} \dfrac{1}{H_1} u v = -\dfrac{1}{H_1} \dfrac{\partial p}{\partial x} + \\ + \dfrac{1}{R_e} \dfrac{1}{H_1^2} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{r' y}{r^2 R_e \sqrt{R_e}} \dfrac{1}{H_1^3} \dfrac{\partial u}{\partial x} + \dfrac{1}{r\sqrt{R_e}} \dfrac{1}{H_1} \dfrac{\partial u}{\partial y} + \\ + \dfrac{2}{r R_e \sqrt{R_e}} \dfrac{1}{H_1^2} \dfrac{\partial v}{\partial x} - \dfrac{1}{r^2 R_e} \dfrac{1}{H_1^2} u - \dfrac{r'}{r^2 R_e \sqrt{R_e}} \dfrac{1}{H_1^3} v \\ \dfrac{1}{R_e} \dfrac{\partial v}{\partial t} + \dfrac{1}{R_e} \dfrac{1}{H_1} u \dfrac{\partial v}{\partial x} + \dfrac{1}{R_e} v \dfrac{\partial v}{\partial y} - \dfrac{1}{\sqrt{R_e}} \dfrac{1}{r H_1} u^2 = \\ = -\dfrac{\partial p}{\partial y} + \dfrac{1}{R_e \sqrt{R_e}} \left(\dfrac{1}{R_e H_1^2} \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{r' y}{r^2 R_e} \dfrac{1}{H_1^3} \dfrac{\partial v}{\partial x} + \right. \\ \left. + \dfrac{1}{r H_1} \dfrac{\partial v}{\partial y} - \dfrac{2}{r H_1^2} \dfrac{\partial u}{\partial x} + \dfrac{1}{r^2 \sqrt{R_e}} v + \dfrac{r'}{r^2} \dfrac{1}{H_1^3} u \right) \end{cases}$$

The flow before the additional motion was defined by these equations (2.7), where however all physical quantities, such as velocity projections (u, v) and pressure (p) could be denoted by indices „s" in order to indicate that they refer to a preceding flow. But, also, (2.7) could be interpreted as equations of the resulting flow after the introduction of the additional motion.

(since $t=T+t_1$, i. e. $t_1>0$). In order to obtain only the equations for the additional flow in a separate form, it is necessary in (2.7) to put

$$u=u_s+u_d,\quad v=v_s+v_d,\quad p=p_s+p_d, \tag{2.8}$$

and to evaluate all terms. Of the equations thus obtained we subtract a system of equations of the form (2.7), in which all quantities should be denoted by indices „s". We obtain equations for a non-steady additional flow around a cylindrical body for an arbitrary Reynolds number:

$$\frac{\partial u_d}{d t_1}+\frac{1}{H_1}\left(u_d\frac{\partial u_s}{\partial x}+u_d\frac{\partial u_d}{\partial x}+u_s\frac{\partial u_d}{\partial x}\right)+v_s\frac{\partial u_d}{\partial y}+v_d\frac{\partial u_s}{\partial y}+v_d\frac{\partial u_d}{\partial y}+$$
$$+\frac{1}{r\sqrt{R_e}}\frac{1}{H_1}(u_s v_d+u_d v_s+u_d v_d)=-\frac{1}{H_1}\frac{\partial p_d}{\partial x}+\frac{1}{R_e}\frac{1}{H_1^2}\frac{\partial^2 u_d}{\partial x^2}+\frac{1}{r\sqrt{R_e}}\frac{1}{H_1}\frac{\partial u_d}{\partial y}+$$
$$+\frac{\partial^2 u_d}{\partial y^2}+\frac{r' y}{r^2 R_e\sqrt{R_e}}\frac{1}{H_1^3}\frac{\partial u_d}{\partial x}+\frac{2}{r R_e\sqrt{R_e}}\frac{1}{H_1^2}\frac{\partial v_d}{\partial x}-\frac{1}{r^2 R_e}\frac{1}{H_1^2}u_d-\frac{r'}{r^2 R_e\sqrt{R_e}}\frac{1}{H_1^3}v_d$$

(2.9)

$$\frac{1}{R_e}\frac{\partial v_d}{\partial t_1}+\frac{1}{R_e}\frac{1}{H_1}\left(u_s\frac{\partial v_d}{\partial x}+u_d\frac{\partial v_s}{\partial x}+u_d\frac{\partial v_d}{\partial x}\right)+\frac{1}{R_e}\left(v_s\frac{\partial v_d}{\partial y}+v_d\frac{\partial v_s}{\partial y}+\right.$$
$$\left.+v_d\frac{\partial v_d}{\partial y}\right)-\frac{1}{\sqrt{R_e}}\frac{1}{r H_1}(2u_s u_d+u_d^2)=-\frac{\partial p_d}{\partial y}+$$
$$+\frac{1}{R_e\sqrt{R_e}}\left(\frac{1}{R_e H_1^2}\frac{\partial^2 v_d}{\partial x^2}+\frac{\partial^2 v_d}{\partial y^2}+\frac{r' y}{r^2 R_e}\frac{1}{H_1^3}\frac{\partial v_d}{\partial x}+\right.$$
$$\left.+\frac{1}{r H_1}\frac{\partial v_d}{\partial y}-\frac{2}{r H_1^2}\frac{\partial u_d}{\partial x}+\frac{1}{r^2\sqrt{R_e}}v_d+\frac{r'}{r^2}\frac{1}{H_1^3}u_d\right).$$

We now consider the flow in the boundary layer on a body, that is, let $R_e\to\infty$, and from (2.9) we obtain equations for the additional boundary layer

$$\frac{\partial u_d}{\partial t_1}+u_s\frac{\partial u_d}{\partial x}+u_d\frac{\partial u_s}{\partial x}+u_d\frac{\partial u_d}{\partial x}+v_s\frac{\partial u_d}{\partial y}+$$
$$+v_d\frac{\partial u_s}{\partial y}+v_d\frac{\partial u_d}{\partial y}=-\frac{1}{\rho}\frac{\partial p_d}{\partial x}+\nu\frac{\partial^2 u_d}{\partial y^2} \tag{2.10}$$
$$0=-\frac{\partial p_d}{\partial y}$$

with the corresponding equation of continuity

$$\frac{\partial u_d}{\partial x}+\frac{\partial v_d}{\partial y}=0. \tag{2.11}$$

The second of the equations (2.10) shows that even now it is possible to assume that the pressure inside the boundary layer is equal to that on the external boundary of the boundary layer, the latter being determined by the external non-viscous flow.

If we adopt Blazius' method of presentation of types of motion of a body through fluid in terms of the external potential velocity, we can arrive at the meaning of the pressure gradient required for the equations (2.10).

Thus, for the case of motion starting from rest we have the well-known Euler's equation

$$\frac{\partial U_s}{\partial t}+U_s\frac{\partial U_s}{\partial x}=-\frac{1}{\rho}\frac{\partial p_s}{\partial x} \tag{2.12}$$

For $t_1>0$, the external potential flow is again defined by Euler's equation. But the preceding potential velocity obtained an increment U_d due to the additional motion. This change in velocity had to be reflected in the change of pressure. Thus we obtain the required pressure „p_d“ :

$$\frac{\partial U_s}{\partial t}+\frac{\partial U_d}{\partial t_1}+(U_s+U_d)\frac{\partial (U_s+U_d)}{\partial x}=-\frac{1}{\rho}\frac{\partial (p_s+p_d)}{\partial x}. \tag{2.13}$$

By substituting (2.12) in (2.13) it is possible to obtain the required pressure gradient:

$$-\frac{1}{\rho}\frac{\partial p_d}{\partial x}=\frac{\partial U_d}{\partial t_1}+U_d\frac{\partial U_d}{\partial x}+U_s\frac{\partial U_d}{\partial x}+U_d\frac{\partial U_s}{\partial x}.$$

If we substitute this relation in (2.10) and taking into account the equation of continuity (2.11) we shall obtain a complete system of equations defining the additional non-steady boundary layer

$$\left\{\begin{aligned}&\frac{\partial u_d}{\partial t}+u_d\frac{\partial u_d}{\partial x}+v_d\frac{\partial u_d}{\partial y}+u_s\frac{\partial u_d}{\partial x}+u_d\frac{\partial u_s}{\partial x}+v_s\frac{\partial u_d}{\partial y}+\\&+v_d\frac{\partial u_s}{\partial y}=\frac{\partial U_d}{\partial t_1}+U_d\frac{\partial U_d}{\partial x}+U_s\frac{\partial U_d}{\partial x}+U_d\frac{\partial U_s}{\partial x}+\nu\frac{\partial^2 u_d}{\partial y^2}\\&\frac{\partial u_d}{\partial x}+\frac{\partial v_d}{\partial y}=0.\end{aligned}\right. \tag{2.14}$$

The boundary and initial conditions are

$$\begin{aligned}&u_d=v_d=0, \quad y=0;\\&u_d=U_d(x,t_1), \quad y=\infty;\\&u_d=v_d=0, \quad \text{for} \quad t_1=0.\end{aligned} \tag{2.15}$$

3. Method for solving additional boundary layer equations. — Let us set up a proces of successive approximations as a method for solving equations (2.15). We form partial equations for individual successive approximations of velocity in the boundary layer. First, we remind ourselves of the principle by means of which Blasius (7) determined the type of motion of the body through a viscous fluid. If the body is started impulsively, Blasius assumes that $U=U(x)$ for $t_1\geqslant 0$, and if the motion is constantly accelerated, then $U=tW(x)$, for $t\geqslant 0$, etc. Here, U is the external potential velocity. Hence, in a coordinate system rigidly connected to the body itself, as in our case here, Blasius assumed that the external potential flow moved in a particular way with respect to the body. This type of motion will be used in the present paper to denote the type of additional motion which takes place with an additional external potential velocity U_d.

Hence, for $t_1=0$, there exists a certain boundary layer (u_s, v_s) on the body. Thereupon, the body is given an additional motion, or, as Blasius put it, as if „an external potential flow U_d of the fluid comes forth". In the initial instants of this additional motion, U_d extends almost completely along longitudinal direction $(u_d \sim U_d)$; actually, to put it more accurately, the preceding lateral motion is weakened by the additional flow (v_d is in opposite direction of v_s): this is represented symbolically by a physical model shown in Figure 2. Since in a space in which the equations of the boundary layer — the latter being almost completely „taken into consideration" by this physical model — are certainly valid,

$$\frac{\partial u_d}{\partial y}>0, \quad \frac{\partial u_s}{\partial y}>0,$$

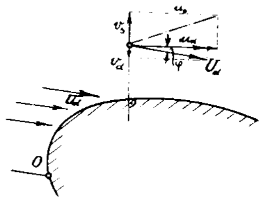

Fig. 2

then from (2.14) it follows that the terms

$$v_s \frac{\partial u_d}{\partial y} \text{ and } \left(v_d \frac{\partial u_s}{\partial y}+v_d \frac{\partial u_d}{\partial y}\right),$$

in effect, cancel each other. Remembering that in any case the lateral velocities in the boundary layer are of a far lower order of magnitude than the longitudinal ones, it is quite reasonable to conclude that the group of convective terms

$$v_s \frac{\partial u_d}{\partial y}+v_d \frac{\partial u_s}{\partial y}+v_d \frac{\partial u_d}{\partial y}$$

from (2.14) are in the initial instants of the additional motion negligibly small compared, for instance, with $\nu \frac{\partial^2 u_d}{\partial y^2}$.

Therefore, if the velocities of the additional boundary layer are represented as sums of certain approximations

$$(3.1) \qquad \begin{aligned} u_d &= u_o(x, y, t_1)+u_1(x, y, t_1) \\ v_d &= v_o(x, y, t_1)+v_1(x, y, t_1) \end{aligned}$$

the elements listed above enable us to derive equations for the first approximation as follows:

$$(3.2) \qquad \left\{ \begin{aligned} &\frac{\partial u_o}{\partial t_1}-\nu \frac{\partial^2 u_o}{\partial y^2}=\frac{\partial U_d}{\partial t_1}+\frac{\partial}{\partial x}\left[U_d(U_s-u_s)\right] \\ &\frac{\partial u_o}{\partial x}+\frac{\partial v_0}{\partial y}=0 \end{aligned} \right.$$

The second equation from (3.2) is the equation of continuity for the first approximation of velocity (u_o, v_o). The boundary conditions are

$$(3.3) \qquad u_o=0, \quad y=0; \quad u_o=U_d(x, t_1), \quad y=\infty .$$

If we use the well-known scheme for deriving equations for successive approximations, it is possible to obtain equations for the second approximation of velocity in the boundary layer

$$(3.4)\qquad \begin{cases} \dfrac{\partial u_1}{\partial t_1} - \nu \dfrac{\partial u_1}{\partial y^2} = U_d \dfrac{\partial U_d}{\partial x} + \dfrac{\partial}{\partial x}(U_d u_s) - (u_s + u_o)\dfrac{\partial u_o}{\partial x} - \\ -(v_s + v_o)\dfrac{\partial u_o}{\partial y} - u_o \dfrac{\partial u_s}{\partial y} - v_o \dfrac{\partial u_s}{\partial y}, \\ \dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = 0 \end{cases}$$

with boundary conditions

$$(3.5)\qquad u_1 = 0, \quad y = 0; \quad u_1 = 0, \quad \text{for} \quad y = \infty.$$

The right-hand sides of equations for the first and second approximations, (3.2) and (3.4) respectively, contain, inter alia, the functions „u_s" and „v_s". These are the well-known Blasius solutions of the preceding boundary layer (7), and with accuracy of the first approximation, for the preceding motion started impulsively, these functions are

$$(3.6)\qquad u_s = U f_1'(\eta) = U\, Erf\, \eta, \quad v_s = -2\sqrt{\nu t}\, U' f_1(\eta),$$

where „f_1" is a universal function which corresponds to the first approximation, and is given in terms of the variable

$$(3.7)\qquad \eta = \frac{y}{2\sqrt{\nu t}}.$$

Thist fact assumes a substantial meaning. Since we prove that for the solution of (3.2) and (3.4) it is suitable to use the new non-steady variable

$$(3.8)\qquad \bar{\eta} = \frac{y}{2\sqrt{\nu t_1}}$$

$$(3.9)\qquad t_1 = t - T$$

we conclude, of course, that functions $f_1'(\eta)$ and $f_1(\eta)$, must first be given in terms of new variable $\bar{\eta}$, and only then to proceed with the solution of equations (3.2) and (3.4).

4. Question of the upper boundary condition. — In non-steady boundary layers there exists a very interesting phenomenon which in the circumstances when it is impossible to avoid the expansion of the function $f_1'(\eta)$ into a series and to retain a finite number of terms — for the purpose of transformation into new „additional" variables — may assume a very important role. Namely, it is well-known that one of the boundary conditions shows that at an infinite distance from the body contour the boundary layer velocity becomes equal to the external potential velocity. Both in case of motion of a cylindrical body from the state of rest and in case of motion of a plane contour from a preceding non-steady motion, this has brought forth the infinite values of the non-steady variable η and $\bar{\eta}$, in one of the boundary conditions.

On the other hand, however, the thickness of the boundary layer is in fact practically limited: we can even say that it is a small quantity (we know that the thickness of the boundary layer is a quantity of order $\frac{l}{\sqrt{R_e}} = \sqrt{\frac{l\nu}{V}}$; if, for instance, we assume that $l = 1$ m, $V = 1$ m/s, $\nu = 0{,}01$ cm/s, for water at 20° C, then we obtain $\sqrt{\frac{l\nu}{V}} = 0{,}1$ cm $= 1$ mm, which proves that the boundary layer thickness is small indeed). Hence, all this leads us to conclude that the value $\eta = \infty$ is more or less of a formal nature, and that it is possible to proceed to the finite, far less values without interfering essentially with the accuracy of the solution. This will be proved by the following calculations.

In the case of a motion started impulsively (8, p. 191) from state of rest, in the first approximation of velocity in the boundary layer when determining the corresponding universal function

$$\zeta_0''' + 2\eta\zeta_0'' = 0$$

$$\zeta_0' = C_1 \int_0^{\eta} e^{-\gamma^2} d\gamma + C_2; \quad \zeta_0'(0) = 0, \quad \zeta_0'(\infty) = 1,$$

we obtained the following values of constants

$$C_1 = 1{,}128, \quad C_2 = 0,$$

and in the second approximation

$$\zeta_1''' + 2\eta\zeta_1'' - 4\zeta_1' = 4(\zeta_0'^2 - \zeta_0\zeta_0'' - 1),$$

$$\zeta_1' = C_1(1 + 2\eta^2) + C_2\left[\frac{\sqrt{\pi}}{2}(1 + 2\eta^2)\,Erf\,\eta + \eta e^{-\eta^2}\right] +$$

$$+ \frac{1}{2}(2\eta^2 - 1)\,Erf^2\eta + \frac{3}{\sqrt{\pi}}\eta e^{-\eta^2} Erf\,\eta + 1 - \frac{4}{3\pi}e^{-\eta^2} + \frac{2}{\pi}e^{-2\eta^2}$$

$$\zeta_1'(0) = 0, \ \zeta_1'(\infty) = 0; \quad C_1 = -1{,}212, \ C_2 = 0{,}804;$$

When the motion is constantly accelerated (8, p. 198), for the first approximation of velocity

$$\zeta_1''' + 2\eta\zeta_1'' - 4\zeta_1' = -4$$

$$\zeta_1' = C_1(1 + 2\eta^2) + C_2\left[\frac{1}{4}(1 + 2\eta^2)(1 - Erf\,\eta) - \frac{1}{2\sqrt{\pi}}\eta e^{-\eta^2}\right] + 1,$$

$$\zeta_1'(0) = 0, \quad \zeta_1'(\infty) = 1,$$

we obtain with boundary conditions given above the following values of constants

$$C_1 = 0, \quad C_2 = -4{,}0.$$

But we shall proceed in accordance with the idea of finity of the boundary layer. We propose to retain for one of the boundary conditions the zero value of the variable ($\eta=0$), and for the second boundary condition we shall take the finite value of the variable „η“, i. e., a certain constant. Instead of $\eta=\infty$, we write equations of boundary conditions with $\eta=2$. Then the constants of integration have the following values: when the motion is started impulsively from rest: $C_1=1{,}133$, $C_2=0$, for the first approximation, and $C_1=-1{,}212$, $C_2=0{,}802$, for the second approximation; when the motion is constantly accelerated from rest: $C_1=0{,}0004$, $C_2=-4{,}0016$. Hence, the difference in the values of constants, and therefore in solutions themselves are very small indeed. In all these cases, the relative discrepancies in values of the constants do not exceed 0,44%.

Hence, in a non-steady boundary layer, the theoretical interval $0\leqslant\eta\leqslant\infty$ can be safely substituted by $0\leqslant\eta\leqslant 2$ with an extraordinary accuracy (the substitution interval can be even $0\leqslant\eta\leqslant 1{,}25$ with a discrepancy of only 7%). This interval of values of the variable η enables us to obtain a satisfactory accuracy with only a few terms of the expansion series when the function „u_s“ is expressed in terms of the new variable.

In addition, since for $\eta=2$, $u=U$, it follows that the value $\eta=2$ represents the thickness of the boundary layer, because this is the only factor in the solutions for the boundary layer which involves the lateral „y“ direction. In view of the identical structure of expressions (3.7) and (3.8), in principle, this numerical value of the non-steady variable could play a role in determining solutions for the additional boundary layer.

It may be noted that this is not the first time that the expression „$\sqrt{\nu t}$“ assumes a designation of an indication of the boundary layer thickness. When he investigated the stability of the velocity profile in a non-steady boundary layer, Schlichting (10) in his „characteristic parameter“ used to take for a quantity proportional to the thickness of the boundary layer the term „$\sqrt{\nu t}$“ (1932).

5. A preceding motion started impulsively. — For the sake of brevity in our discussions we limit ourselves only to the first approximations of the preceding and additional boundary layers. In that case, as seen from (3.2), it is necessary to adapt the function „u_s“ to the new variables (3.8) and (3.9) in the following way:

$$u_s = U f_1' = U\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\gamma^2}\,d\gamma$$

In fact, the function

$$f_1' = \frac{2}{\sqrt{\pi}}\int_0^{\bar{\eta}\sqrt{\frac{\tau}{1+\tau}}} e^{-\gamma^2}\,d\gamma \tag{5.1}$$

should be expressed in terms of „$\bar{\eta}$“ and „τ“ in a suitable form. Here, „τ“ denotes non-dimensional time of duration of the additional boundary layer

$$\tau=\frac{t_1}{T} \tag{5.2}$$

Since it is possible to express the integral function (3.10) by a uniformly convergent series

$$e^{-\gamma^2} = 1 - \frac{\gamma^2}{1!} + \frac{\gamma^4}{2!} - \frac{\gamma^6}{3!} + \cdots$$

in any finite interval, then also the definite integral (5.1) by term-by-term integration can be represented by a convergent series and worked out with any degree of accuracy. This statement, which is well known from the theory of series, finds especially useful application in the present paper, since the nature of this convergent series is such that only several initial terms can provide good accuracy. For the purpose of illustration, let it be noted that the series can be worked out to accuracy of 0,0001 with only the first three terms.

Thus, by retaining only the first two terms of the series, and following certain transformations (9), it is possible to obtain the relation

$$(5.3) \qquad f_1' \approx \frac{8}{15\sqrt{\pi}}\,\bar{\eta} + \left(\frac{8}{5\sqrt{\pi}}\,\bar{\eta} - \frac{8}{45\sqrt{\pi}}\,\bar{\eta}^3\right)\frac{t_1}{t} - \frac{8}{15\sqrt{\pi}}\,\bar{\eta}^3\left(\frac{t_1}{t}\right)^2$$

in which factors $\left(\frac{t_1}{t}\right)$ and $\left(\frac{t_1}{t}\right)^2$ should be expressed in terms of „t_1" and „τ", respectively, by means of a polynomial in order to enable the equation (3.2) to be solved analytically. A discussion in greater detail (9) has been devoted to the rate of increase of the degree of accuracy of the approximate expression with the raised order of the approximative series ($P_i(\tau)$:

$$\frac{t_1}{t} = \frac{\tau}{1+\tau} \approx P_i(\tau)$$

as seen from Figure 3. The conclusion reached is that for

$$0 < \tau \leqslant 0{,}6$$

satisfactory accuracy is obtained by the 6th degree polynomial $P_6(\tau)$. Fig. 4 forcefully proves that the approximative expressions which correspond to the present discussion:

$$(5.4) \qquad \begin{cases} \dfrac{t_1}{t} \approx \tau - \tau^2 + \tau^3 - \tau^4 + \tau^5 - \tau^6 \\[2ex] \left(\dfrac{t_1}{t}\right)^2 \approx \tau^2 - 2\,\tau^3 + 3\,\tau^4 - 4\,\tau^5 + 3\,\tau^6 \end{cases}$$

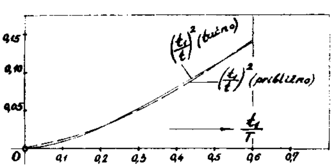

Fig. 3

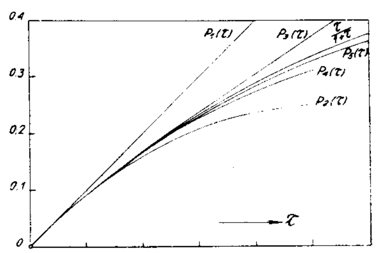

Fig. 4

If (5.4) are substituted in (5.3), we obtain finally the function „u_s“ adapted to the new variable:

$$\frac{u_s}{U_s} = \sum_{k=0}^{6} \omega_k(\bar{\eta})\,\tau^k \tag{5.5}$$

where the coefficients $\omega_k(\bar{\eta})$ are the already known functions

$$\left\{\begin{aligned} &\omega_0 = \frac{8}{15\sqrt{\pi}}\bar{\eta}, \quad \omega_1 = \frac{5}{5\sqrt{\pi}}\left(\bar{\eta} - \frac{1}{9}\bar{\eta}^{3}\right)^{*}, \quad \omega_2 = -\frac{8}{5\sqrt{\pi}}\left(\bar{\eta} + \frac{2}{9}\bar{\eta}^3\right), \\ &\omega^3 = \frac{8}{\sqrt{\pi}}\left(\frac{1}{5}\bar{\eta} + \frac{1}{9}\bar{\eta}^3\right), \quad \omega_4 = -\frac{8}{5\sqrt{\pi}}\left(\bar{\eta} + \frac{8}{9}\bar{\eta}^3\right), \\ &\omega^5 = \frac{8}{5\sqrt{\pi}}\left(\bar{\eta} + \frac{11}{9}\bar{\eta}^3\right), \quad \omega^6 = -\frac{8}{5\sqrt{\pi}}\left(\bar{\eta} + \frac{8}{9}\bar{\eta}^3\right). \end{aligned}\right. \tag{5.6}$$

The satisfactory accuracy of the expression (5.5) can also be checked by means of Fig. 5 in which we observe a discrepancy of the approximate expression $\left(\frac{u_s}{U}\right)_{\bar{\eta}}$ from the accurate one $\left(\frac{u_s}{U}\right)_{\eta}$. It is possible to obtain an improvement inaccuracy (9) of the adapted expression $\left(\frac{u_s}{U}\right)_{\bar{\eta}}$.

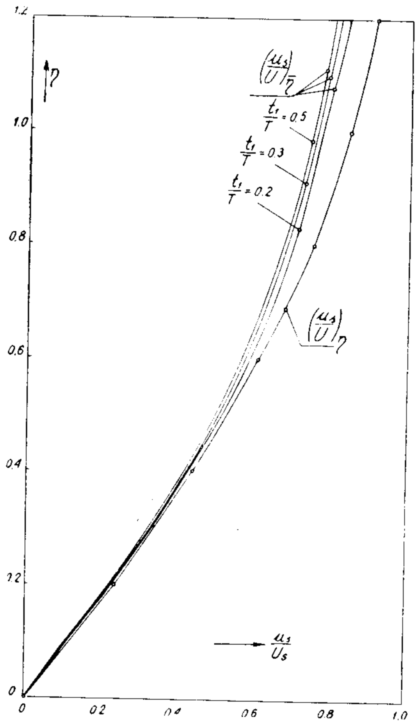

Fig. 5

6. Preceding motion started impulsively, additional motion started impulsively. — A cylindrical body is started by an impulse ($U_s = U(x)$) normal to the direction of its generatrices. At the instant $t = T$, the same body is given an additional impulse of the same direction ($U_d = U(x)$). If these values and the expression (5.5) are substituted in (3.2), we obtain

$$\begin{aligned} \frac{\partial u_0}{\partial t_1} - \nu\frac{\partial^2 u_0}{\partial y^2} &= 2\,UU'(1-\omega_0) - t_1\frac{2\,UU'}{T_1}\omega_1 - \\ &- t_1^2\frac{2\,UU'}{T^2}\omega_2 - t_1^3\frac{2\,UU'}{T^3}\omega_3 - \\ &- t_1^4\frac{2\,UU'}{T^4}\omega_4 - t_1^5\frac{2\,UU'}{T^5}\omega_5 - t_1^6\frac{2\,UU'}{T^6}\omega_6. \end{aligned} \tag{6.1}$$

If we assume that the solution of the preceding equation is in the form of

$$\begin{aligned} u_0 &= U\zeta_0'(\bar{\eta}) + t_1\,2\,UU'\zeta_1'(\bar{\eta}) + t_1^2\frac{2\,UU'}{T}\zeta_2'(\bar{\eta}) + t_1^3\frac{2\,UU'}{T^2}\zeta_3'(\bar{\eta}) + \\ &+ t_1^4\frac{2\,UU'}{T^3}\zeta_4'(\bar{\eta}) + t_1^5\frac{2\,UU'}{T^4}\zeta_5'(\bar{\eta}) + t_1^6\frac{2\,UU'}{T^5}\zeta_6'(\bar{\eta}) + t_1^7\frac{2\,UU'}{T^6}\zeta_7'(\bar{\eta}) \end{aligned} \tag{6.2}$$

* The symbol $\bar{\eta}^3$, and all others of similar kind, signify $\bar{\bar{\eta}}^{3}$

we shall obtain for the unknown coefficients of the universal function in terms of $\bar{\eta}$, the following ordinary differential equations

$$
(6.3)\qquad
\begin{cases}
\zeta_0''' + 2\bar{\eta}\,\zeta_0'' = 0 \\
\zeta_1''' + 2\bar{\eta}\,\zeta_1'' - 4\,\zeta_1' = 4\,(\omega_0 - 1) \\
\zeta_2''' + 2\bar{\eta}\,\zeta_2'' - 8\,\zeta_2' = 4\,\omega_1 \\
\zeta_3''' + 2\bar{\eta}\,\zeta_3'' - 12\,\zeta_3' = 4\,\omega_2 \\
\zeta_4''' + 2\bar{\eta}\,\zeta_4'' - 16\,\zeta_4' = 4\,\omega_3 \\
\zeta_5''' + 2\bar{\eta}\,\zeta_5'' - 20\,\zeta_5' = 4\,\omega_4 \\
\zeta_6''' + 2\bar{\eta}\,\zeta_6'' - 24\,\zeta_6' = 4\,\omega_5 \\
\zeta_7''' + 2\bar{\eta}\,\zeta_7'' - 28\,\zeta_7' = 4\,\omega_6
\end{cases}
$$

The righ-hand sides of the equations (6.3) are the already known functions (5.6).

The fact that the left hand sides of all the equations (6.3) can be taken into account by a general from

$$y'' + 2\,\eta\,y' - 4\,\alpha\,y = 0; \quad \alpha = 0,1,2,\ldots,\ 7.$$

will contribute to the solution of the equations (6.3).

By introducing suitable substitutions of the function and independently variable from (6.4), the latter can be reduced to Weber's functions, or the so-called parabolyc cylinder functions. If in these transformations we start from the integral representation of the parabolyc function, using again suitable substitutions of the variable, it is possible to show that the solutions of (6.4) for $\alpha \geqslant 0$, as in our case, are indeed reduced to Gauss' error function:

$$g_\alpha(\eta) = \frac{2}{\sqrt{\pi}\,\Gamma(2\,\alpha+1)} \int_{\eta}^{\infty} (\gamma - \eta)^{2\alpha}\, e^{-\gamma^2}\, d\gamma$$

Hence, it is possible to find the particular integrals of certain homogeneous parts of the equations (6.3) only by introducing the corresponding values of the parameter

The solutions of the differential equations (6.3) are as follows:

$$\zeta_0' = C_1 \int_0^{\bar{\eta}} e^{-\gamma^2}\, d\gamma + C_2; \quad C_1 = \frac{2}{\sqrt{\pi}}, \quad C_2 = 0;$$

$$\zeta_1' = C_1\,(1 + 2\,\bar{\eta}^2) + C_2 \left[\frac{1}{4}(1 + 2\,\bar{\eta}^2)\, Erf\,\bar{\eta} + \frac{1}{2\sqrt{\pi}}\,\bar{\eta}\, e^{-\bar{\eta}^2} \right] - \frac{16}{15\sqrt{\pi}}\,\bar{\eta} + 1$$

$$C_1 = -1, \quad C_2 = 4{,}091;$$

$$\zeta_2' = C_1 \left(1 + 4\,\bar{\eta}^2 + \frac{4}{3}\,\bar{\eta}^4\right) + C_2 \left[\frac{1}{32}\left(1 + 4\,\bar{\eta}^2 + \frac{4}{3}\,\bar{\eta}^4\right) Erf\,\bar{\eta} + \frac{1}{24\sqrt{\pi}} \left(\bar{\eta}^3 + \right.\right.$$

$$+\frac{5}{2}\bar{\eta}\Big)e^{-\bar{\eta}^2}\Big]+\frac{16}{45\sqrt{\pi}}\bar{\eta}^3-\frac{32}{45\sqrt{\pi}}\bar{\eta};\quad C_1=0,\quad C_2=-0{,}670;$$

$$\zeta_3'=C_1\left(1+6\bar{\eta}^2+4\bar{\eta}^4+\frac{8}{15}\bar{\eta}^6\right)+C_2\left[\frac{1}{384}\left(1+6\bar{\eta}^2+4\bar{\eta}^4+\frac{8}{15}\bar{\eta}^6\right)Erf\,\bar{\eta}+\right.$$

$$\left.+\frac{1}{720\sqrt{\pi}}\left(\bar{\eta}^5+7\bar{\eta}^3+\frac{33}{4}\bar{\eta}\right)e^{-\bar{\eta}^2}\right]+\frac{32}{135\sqrt{\pi}}\bar{\eta}^3+\frac{176}{225\sqrt{\pi}}\bar{\eta};\ C_1=0,\ C_2=-6{,}09;$$

$$\zeta_4'=C_1\left(1+8\bar{\eta}^2+8\bar{\eta}^4+\frac{32}{15}\bar{\eta}^6+\frac{16}{105}\bar{\eta}^8\right)+C_2\left[\frac{1}{6144}\left(1+8\bar{\eta}^2+\right.\right.$$

$$\left.+8\bar{\eta}^4+\frac{32}{15}\bar{\eta}^6+\frac{16}{105}\bar{\eta}^8\right)Erf\,\bar{\eta}+\frac{1}{40320}\left(\frac{1}{2}\bar{\eta}^7+\frac{27}{4}\bar{\eta}^5+\frac{185}{8}\bar{\eta}^3+\right.$$

$$\left.\left.+\frac{279}{16}\bar{\eta}\right)\frac{2}{\sqrt{\pi}}e^{-\bar{\eta}^2}\right]-\frac{16}{45\sqrt{\pi}}\bar{\eta}^3-\frac{64}{105\sqrt{\pi}}\bar{\eta};\quad C_1=0,\quad C_2=41{,}875;$$

$$\zeta_5'=C_1\left(1+10\bar{\eta}^2+\frac{40}{3}\bar{\eta}^4+\frac{16}{3}\bar{\eta}^6+\frac{16}{21}\bar{\eta}^8+\frac{32}{945}\eta^{10}\right)+$$

$$+C_2\left[\frac{1}{122880}\left(1+10\bar{\eta}^2+\frac{40}{3}\bar{\eta}^4+\frac{16}{3}\bar{\eta}^6+\frac{16}{21}\bar{\eta}^8+\frac{32}{945}\bar{\eta}^{10}\right)Erf\,\bar{\eta}+\right.$$

$$\left.+\frac{1}{3628800}\left(\frac{1}{2}\bar{\eta}^9+11\bar{\eta}^7+\frac{147}{2}\bar{\eta}^5+165\bar{\eta}^3+\frac{2895}{32}\bar{\eta}\right)\frac{2}{\sqrt{\pi}}e^{-\bar{\eta}^2}\right]+$$

$$+\frac{128}{315\sqrt{\pi}}\bar{\eta}^3+\frac{464}{945\sqrt{\pi}}\bar{\eta};\quad C_1=0,\quad C_2=-356{,}119;$$

$$\zeta_6=C_1\left(1+12\bar{\eta}^2+20\bar{\eta}^4+\frac{32}{3}\bar{\eta}^6+\frac{16}{7}\bar{\eta}^8+\frac{64}{315}\bar{\eta}^{10}+\frac{64}{10395}\bar{\eta}^{12}\right)+$$

$$+C_2\left[\frac{1}{2949120}\left(1+12\bar{\eta}^2+20\bar{\eta}^4+\frac{32}{3}\bar{\eta}^6+\frac{16}{7}\bar{\eta}^8+\frac{64}{315}\bar{\eta}^{10}+\right.\right.$$

$$\left.+\frac{64}{10395}\bar{\eta}^{12}\right)Erf\,\bar{\eta}+\frac{1}{479001600}\left(\frac{1}{2}\bar{\eta}^{11}+\frac{65}{4}\bar{\eta}^9+\frac{711}{4}\bar{\eta}^7+\right.$$

$$\left.\left.+\frac{6279}{8}\bar{\eta}^5+\frac{41685}{32}\bar{\eta}^3+\frac{35685}{64}\bar{\eta}\right)\frac{2}{\sqrt{\pi}}e^{-\bar{\eta}^2}\right]-\frac{176}{405\sqrt{\pi}}\bar{\eta}^3-\frac{608}{1485\sqrt{\pi}}\bar{\eta};$$

$$C_1=0,\quad C_2=3845{,}40;$$

$$\zeta_7'=C_1\left(1+14\bar{\eta}^2+28\bar{\eta}^4+\frac{56}{3}\bar{\eta}^6+\frac{16}{3}\bar{\eta}^8+\frac{32}{45}\bar{\eta}^{10}+\frac{64}{1485}\bar{\eta}^{12}+\right.$$

$$\left.+\frac{128}{135135}\bar{\eta}^{14}\right)+C_2\left[\frac{1}{82575360}\left(1+14\bar{\eta}^2+28\bar{\eta}^4+\frac{56}{3}\bar{\eta}^6+\frac{16}{3}\bar{\eta}^8+\right.\right.$$

$$+\frac{32}{45}\bar{\eta}^{10}+\frac{64}{1485}\bar{\eta}^{12}+\frac{128}{135135}\bar{\eta}^{14}\Big) Erf\,\bar{\eta}+\frac{1}{87178291200}\Big(\frac{1}{2}\bar{\eta}^{13}+$$

$$+\frac{45}{2}\bar{\eta}^{11}+\frac{2915}{8}\bar{\eta}^{9}+\frac{10575}{4}\bar{\eta}^{7}+\frac{278019}{32}\bar{\eta}^{5}+$$

$$+\frac{364665}{32}\bar{\eta}^{3}+\frac{509985}{128}\bar{\eta}\Big)\frac{2}{\sqrt{\pi}}e^{-\bar{\eta}^2}\Big]+\frac{128}{495\sqrt{\pi}}\bar{\eta}^{3}+\frac{656}{214\sqrt{\pi}}\bar{\eta};$$

$$C_1=0,\quad C_2=-31363{,}070.$$

In order to find the above given values of the constants, the following boundary conditions were used

(6.5) $$\zeta_0'(0)=0,\quad \zeta_0'(\infty)=1,$$

(6.6) $$\zeta_i'(0)=0,\quad \zeta_i'(2)=0,\quad \text{for}\quad i=1,2,\ldots,7.$$

It should be emphasized that this property of the variable $\bar{\eta}$ to have two values is only of a formal character. As shown already, if instead of $\bar{\eta}=\infty$, we assume $\bar{\eta}=2$, in the boundary level, a minimum error of only 0.44% is made in the values of the constants of integration. The fact that the universal function $\zeta_0'(\bar{\eta})$ is a factor adjacent to $U(x)$ in (6.2), and keeping in mind (6.5), is of some importance because it makes, at least formally, that the additional boundary layer also has an asymptotic character. The universal functions we found as the solutions of the differential equations (6.3) are shown in Figure 6.

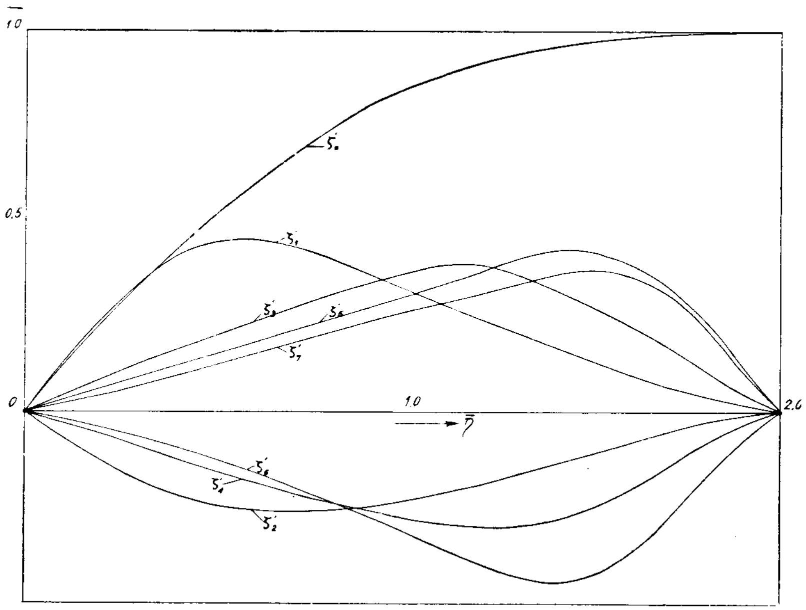

Fig. 6

In accordance with the basic conception of the method used to solve the entire problem, this conception being analytically written in the form of the relations (2.8), the total velocity in the boundary layer, in this sense, is

$$u = U f_1'(\eta) + U [\zeta_0'(\bar{\eta}) + t_1\, 2U' \zeta_1'(\bar{\eta}) + t_1^2 \frac{2U'}{T} \zeta_2'(\bar{\eta}) + t_1^3 \frac{2U'}{T^2} \zeta_3'(\bar{\eta}) +$$

(6.7)

$$+ t_1^4 \frac{2U'}{T^3} \zeta_4'(\bar{\eta}) + t_1^5 \frac{2U'}{T^4} \zeta_5'(\bar{\eta}) + t_1^6 \frac{2U'}{T^5} \zeta_6'(\bar{\eta}) + t_1^7 \frac{2U'}{T^6} \zeta_7'(\bar{\eta}).$$

Using the condition of separation of boundary layer $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ we obtain the following equation

$$\frac{4}{15} f_1''(0) + \zeta_0''(0) + t_1 \left[\frac{4}{5} \frac{f_1''(0)}{T} + 2U' \zeta_1''(0)\right] + t_1^2 \left[2U' \frac{\zeta_2''(0)}{T} - \frac{4}{5} \frac{f_1''(0)}{T^2}\right] +$$

(6.8)

$$+ t_1^3 \left[2U' \frac{\zeta_3''(0)}{T^2} + \frac{4}{5} \frac{f_1''(0)}{T^3}\right] + t_1^4 \left[2U' \frac{\zeta_4''(0)}{T^3} - \frac{4}{5} \frac{f_1''(0)}{T^4}\right] +$$

$$+ t_1^5 \left[2U' \frac{\zeta_5''(0)}{T^4} + \frac{4}{5} \frac{f_1''(0)}{T^5}\right] + t_1^6 \left[2U' \frac{\zeta_6''(0)}{T^5} - \frac{4}{5} \frac{f_1''(0)}{T^6}\right] + t_1^7\, 2U' \frac{\zeta_7''(0)}{T^6} = 0.$$

Thence it is possible to work out the instant of separation of the boundary layer at any point along the contour of the cylindrical body for a given value of the constant.

Example: A circular cylinder, having a radius $R = 50$ cm is started by an impulse $U_\infty = 10$ cm/s; then, at $T = 3/2$ sec, the cylinder is given an additional impulse $U_\infty = 10$ cm/s. Determine when the boundary layer will become separated in the rear stagnation point.

First we check whether the boundary layer became already separated in the rear stagnation point due to the preceding impulse. By using Blasius' solution: $t_s = 0{,}351 \frac{R}{U_\infty}$, we learn that the separation would take place at that point at $t_s = 1{,}755$ sec, which is greater than the assumed value of $T = \frac{3}{2}$ sec. Since for the circular cylinder the potential velocity is $U = 2U_\infty \sin \frac{x}{R}$ for $\frac{x}{R} = \pi$, we shall have $U' = -\frac{2}{5}$. If this is substituted, complete with zero values of the second derivatives of the corresponding universal functions, in (6.8), we have

$$0{,}045\, t_1^7 + 0{,}208\, t_1^6 - 0{,}290\, t_1^5 + 0{,}388\, t_1^4 - 0{,}490\, t_1^3 +$$

(6.9)

$$+ 0{,}574\, t_1^2 + 2{,}861\, t_1 - 5{,}358 = 0.$$

The solution of this equation

(6.10) $$t_1 = 1{,}35 \text{ sec}$$

is an indication of the length of time that elapses after an additional impulse is given until the separation of the boundary layer in the rear stagnation point takes place. Thus, instead of the boundary layer becoming separated after 0.255 sec if there is an additional impulse, the separation of the boundary layer takes place considerably later — in fact, 1.30 seconds after the motion was given an additional impulse.

By means of (6.7) it is possible to calculate the boundary layer velocity at a specified position of the contour for different instants, in the first case, or different positions on the body's contour for a specified instant in the

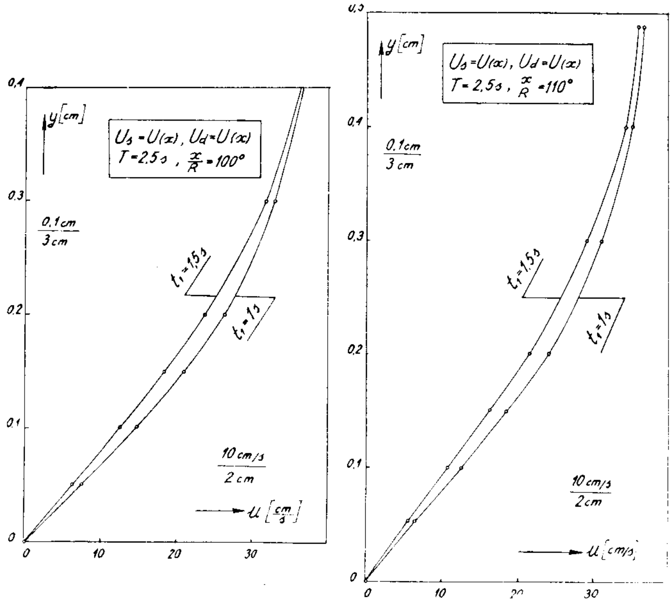

Fig. 7

second case. The investigations in these two directions of approach have carried out in great detail (9), accompanied by an extensive theoretical and tabular analysis. Here we give only the corresponding diagrams.

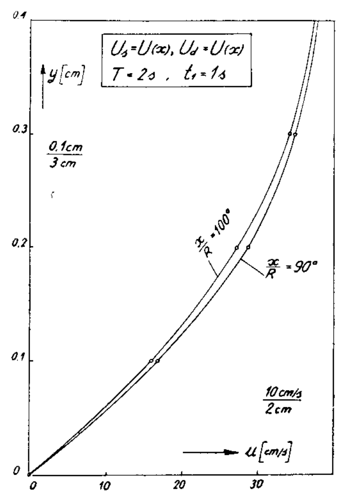

Fig. 8

The results of investigations involving the first case above (Fig. 7) point to a natural development of the boundary layer velocity in the course of time, suggesting the possibility of separation of the boundary layer at that particular position of the cylinder's contour. The profiles of boundary velocities have anticipated forms. A gradual „deterioration" of the velocity profile while nearing the zone of the first separation of the boundary layer — is an obvious fact here.

7. Preceding motion started impulsively, additional motion accelerated constantly. — External potential velocities of these two motions have the following values:

$$U_s = U(x), \quad U_d = t_1 W(x).$$

If these values are substituted in (5.5) and in the equation (3.2), we obtain

$$\frac{\partial u_0}{\partial t_1} - \nu \frac{\partial^2 u_0}{\partial y^2} = W + t_1 F(1-\omega_0) - t_1^2 \frac{F}{T}\omega_1 - t_1^3 \frac{F}{T^2}\omega_2 - \tag{7.1}$$

$$- t_1^4 \frac{F}{T^3}\omega_3 - t_1^5 \frac{F}{T^4}\omega_4 - t_1^6 \frac{F}{T^5}\omega_5 - t_1^7 \frac{F}{T^6}\omega_6$$

where a shorter symbol $F = UW' + U'W$ has been introduced.

If the solution to this equation is sought in the form of

$$u_0 = t_1 W \zeta_0'(\bar\eta) + t_1^2 F \zeta_1'(\bar\eta) + t_1^3 \frac{F}{T}\zeta_2'(\bar\eta) + t_1^4 \frac{F}{T^2}\zeta_3'(\bar\eta) + \tag{7.2}$$

$$+ t_1^5 \frac{F}{T^3}\zeta_4'(\bar\eta) + t_1^6 \frac{F}{T^4}\zeta_5'(\bar\eta) + t_1^7 \frac{F}{T^5}\zeta_6'(\bar\eta) + t_1^8 \frac{F}{T^6}\zeta_7'(\bar\eta)$$

we obtain for the unknown functions of the variable $\bar\eta$, the following relations

$$\left\{\begin{array}{l}
\zeta_0''' + 2\bar\eta\zeta_0'' - 4\zeta_0' = -4 \\
\zeta_1''' + 2\bar\eta\zeta_1'' - 8\zeta_1' = 4(\omega_{0-1}) \\
\zeta_2''' + 2\bar\eta\zeta_2'' - 12\zeta_2' = 4\,\omega_1 \\
\zeta_3''' + 2\bar\eta\zeta_3'' - 16\zeta_3' = 4\,\omega_2 \\
\zeta_4''' + 2\bar\eta\zeta_4'' - 20\zeta_4' = 4\,\omega_3 \\
\zeta_5''' + 2\bar\eta\zeta_5'' - 24\zeta_5' = 4\,\omega_4 \\
\zeta_6''' + 2\bar\eta\zeta_6'' - 28\zeta_6' = 4\,\omega_5 \\
\zeta_7''' + 2\bar\eta\zeta_7'' - 32\zeta_7' = 4\,\omega_6
\end{array}\right. \tag{7.3}$$

The solutions of these second order linear non-homogeneous differential equations are as follows:

$$\zeta_0' = C_1(1+2\bar\eta^2) + C_2\left[\frac{1}{4}(1+2\bar\eta^2)(1-Erf\bar\eta) - \frac{1}{2\sqrt{\pi}}\bar\eta e^{-\bar\eta^2}\right] + 1;$$

$$C_1 = 0, \quad C_2 = -4;$$

$$\zeta_1' = C_1\left(1 + 4\bar\eta^2 + \frac{4}{3}\bar\eta^4\right) + C_2\left[\frac{1}{32}\left(1 + 4\bar\eta^2 + \frac{4}{3}\bar\eta^4\right)Erf\bar\eta + \frac{1}{24\sqrt{\pi}}\left(\bar\eta^3 + \right.\right.$$

$$\left.\left. + \frac{5}{2}\bar\eta\right)e^{-\bar\eta^2}\right] - \frac{16}{45\sqrt{\pi}}\bar\eta + \frac{1}{2}; \quad C_1 = 7.10^{-5}, \quad C_2 = -16{,}028;$$

$$\zeta_2' = C_1\left(1+6\bar\eta^2+4\bar\eta^4+\frac{8}{15}\bar\eta^6\right)+C_2\left[\frac{1}{384}\left(1+6\bar\eta^2+4\bar\eta^4+\frac{8}{15}\bar\eta^6\right)Erf\,\bar\eta+\right.$$

$$\left.+\frac{1}{720\sqrt{\pi}}\left(\bar\eta^5+7\bar\eta^3+\frac{33}{4}\bar\eta\right)e^{-\bar\eta^2}\right]+\frac{16}{135\sqrt{\pi}}\bar\eta^3-\frac{128}{225\sqrt{\pi}}\bar\eta;$$

$$C_1 = 87\cdot 10^{-5}, \quad C_2 = -0{,}336;$$

$$\zeta_3' = C_1\left(1+8\bar\eta^2+8\bar\eta^4+\frac{32}{15}\bar\eta^6+\frac{16}{105}\bar\eta^8\right)+C_2\left[\frac{1}{6144}\left(1+8\bar\eta^2+8\bar\eta^4+\right.\right.$$

$$\left.\left.+\frac{32}{15}\bar\eta^6+\frac{16}{105}\bar\eta^8\right)Erf\,\bar\eta+\frac{1}{40320}\left(\frac{1}{2}\bar\eta^7+\frac{27}{4}\bar\eta^5+\frac{185}{8}\bar\eta^3+\frac{279}{16}\bar\eta\right)\frac{2}{\sqrt{\pi}}e^{-\bar\eta^2}\right]+$$

$$+\frac{32}{225\sqrt{\pi}}\bar\eta^3+\frac{272}{525\sqrt{\pi}}\bar\eta; \quad C_1 = -364\cdot 10^{-5}; \quad C_2 = 22{,}777;$$

$$\zeta_4' = C_1\left(1+10\bar\eta^2+\frac{40}{3}\bar\eta^4+\frac{16}{3}\bar\eta^6+\frac{16}{21}\bar\eta^8+\frac{32}{945}\bar\eta^{10}\right)+$$

$$+C_2\left[\frac{1}{122880}\left(1+10\bar\eta^2+\frac{40}{3}\bar\eta^4+\frac{16}{3}\bar\eta^6+\frac{16}{21}\bar\eta^8+\frac{32}{945}\bar\eta^{10}\right)Erf\,\bar\eta+\right.$$

$$\left.+\frac{1}{3628800}\left(\frac{1}{2}\bar\eta^9+11\bar\eta^7+\frac{147}{2}\bar\eta^5+165\bar\eta^3+\frac{2895}{32}\bar\eta\right)\frac{2}{\sqrt{\pi}}e^{-\bar\eta^2}\right]-$$

$$-\frac{16}{63\sqrt{\pi}}\bar\eta^3-\frac{416}{945\sqrt{\pi}}\bar\eta; \quad C_1 = 0, \quad C_2 = 244{,}776;$$

$$\zeta_5' = C_1\left(1+12\bar\eta^2+20\bar\eta^4+\frac{32}{3}\bar\eta^6+\frac{16}{7}\bar\eta^8+\frac{64}{315}\bar\eta^{10}+\frac{64}{10395}\bar\eta^{12}\right)+$$

$$+C_2\left[\frac{1}{2949120}\left(1+12\bar\eta^2+20\bar\eta^4+\frac{32}{3}\bar\eta^6+\frac{16}{7}\bar\eta^8+\frac{64}{315}\bar\eta^{10}+\frac{64}{10395}\bar\eta^{12}\right)Erf\,\bar\eta+\right.$$

$$+\frac{1}{479001600}\left(\frac{1}{2}\bar\eta^{11}+\frac{65}{4}\bar\eta^9+\frac{711}{4}\bar\eta^7+\frac{6279}{8}\bar\eta^5+\frac{41685}{32}\bar\eta^3+\right.$$

$$\left.\left.+\frac{35685}{64}\bar\eta\right)\frac{2}{\sqrt{\pi}}e^{-\bar\eta^2}\right]+\frac{128}{405\sqrt{\pi}}\bar\eta^3+\frac{112}{297\sqrt{\pi}}\bar\eta;$$

$$C_1 = 0, \quad C_2 = -2935{,}550$$

$$\zeta_6' = C_1\left(1+14\bar\eta^2+28\bar\eta^4+\frac{56}{3}\bar\eta^6+\frac{16}{3}\bar\eta^8+\frac{32}{45}\bar\eta^{10}+\frac{64}{1485}\bar\eta^{12}+\right.$$

$$\left.+\frac{128}{135135}\bar\eta^{14}\right)+C_2\left[\frac{1}{82575360}\left(1+14\bar\eta^2+28\bar\eta^4+\frac{56}{3}\bar\eta^6+\frac{32}{6}\bar\eta^8+\right.\right.$$

$$+\frac{32}{45}\bar{\eta}^{10}+\frac{64}{1485}\bar{\eta}^{12}+\frac{128}{135135}\bar{\eta}^{14}\Big)Erf\,\bar{\eta}+\frac{1}{87178291200}\Big(\frac{1}{2}\bar{\eta}^{13}+\frac{45}{2}\bar{\eta}^{11}+$$

$$+\frac{2915}{8}\bar{\eta}^{9}+\frac{10575}{4}\bar{\eta}^{7}+\frac{278019}{32}\bar{\eta}^{5}+\frac{364665}{32}\bar{\eta}^{3}+\frac{509985}{128}\bar{\eta}\Big)\frac{2}{\sqrt{\pi}}e^{-\bar{\eta}^2}\Big]-$$

$$-\frac{16}{45\sqrt{\pi}}\bar{\eta}^{3}-\frac{64}{195\sqrt{\pi}}\bar{\eta};\quad C_1=0,\quad C_2=41125{,}0;$$

$$\zeta_7'=C_1\Big(1+16\bar{\eta}^2+\frac{112}{3}\bar{\eta}^4+\frac{448}{15}\bar{\eta}^6+\frac{32}{3}\bar{\eta}^8+$$

$$+\frac{256}{135}\bar{\eta}^{10}+\frac{256}{1485}\bar{\eta}^{12}+\frac{1024}{135135}\bar{\eta}^{14}+\frac{256}{2027025}\bar{\eta}^{16}\Big)+$$

$$+C_2\Big[\frac{1}{2642411520}\Big(1+16\bar{\eta}^2+\frac{112}{3}\bar{\eta}^4+\frac{448}{15}\bar{\eta}^6+\frac{32}{3}\bar{\eta}^8+$$

$$+\frac{256}{135}\bar{\eta}^{10}+\frac{256}{1485}\bar{\eta}^{12}+\frac{1024}{135135}\bar{\eta}^{14}+\frac{256}{2027025}\bar{\eta}^{16}\Big)Erf\,\bar{\eta}+$$

$$+\frac{1}{20922789888000}\Big(\frac{1}{2}\bar{\eta}^{15}+\frac{119}{3}\bar{\eta}^{13}+\frac{5343}{8}\bar{\eta}^{11}+$$

$$+\frac{115005}{16}\bar{\eta}^{9}+\frac{1245915}{32}\bar{\eta}^{7}+\frac{6506325}{64}\bar{\eta}^{5}+$$

$$+\frac{14073885}{128}\bar{\eta}^{3}+\frac{8294895}{256}\bar{\eta}\Big)\frac{2}{\sqrt{\pi}}e^{-\bar{\eta}^2}\Big]+\frac{128}{585\sqrt{\pi}}\bar{\eta}^{3}+\frac{752}{2925\sqrt{\pi}}\bar{\eta}.$$

$$C_1=0,\quad C_2=-416297{,}260.$$

The constants of integrations are determined from the boundary conditions:

$$\zeta_0'(0)=0,\quad \zeta_0'(\infty)=1,$$

$$\zeta_i'(0)=0,\quad \zeta_i'(2)=0,\quad i=1,2,\ldots,\ 7.$$

Here we could repeat all the conclusions we made already in the preceding section 6.

The total velocity in the boundary layer is

$$u=Uf_1'(\bar{\eta})+t_1W\zeta_0'(\bar{\eta})+t_1^2F\zeta_1'(\bar{\eta})+t_1^3\frac{F}{T}\zeta_2'(\bar{\eta})+t_1^4\frac{F}{T^2}\zeta_3'(\bar{\eta})+$$

(7.4)

$$+t_1^5\frac{F}{T^3}\zeta_4'(\bar{\eta})+t_1^6\frac{F}{T^4}\zeta_5'(\bar{\eta})+t_1^7\frac{F}{T^5}\zeta_6'(\bar{\eta})+t_1^8\frac{F}{T^6}\zeta_7'(\bar{\eta}),$$

and from the separation condition $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$, follows

$$
\begin{aligned}
(7.5)\qquad & t_1^8 \frac{\zeta_7''(0)}{T^6} + t_1^7 \frac{\zeta_6''(0)}{T^5} + t_1^6 \left[\frac{\zeta_5''(0)}{T^4} + \frac{f_1''(0)}{T^6}\right] + t_1^5 \left[\frac{f_4''(0)}{T^3} - \frac{f_1''(0)}{T^5}\right] + \\
& + t_1^4 \left[\frac{\zeta_3''(0)}{T^2} + \frac{f_1''(0)}{T^4}\right] + t_1^3 \left[\frac{\zeta_2''(0)}{T} - \frac{f_1''(0)}{T^3}\right] + t_1^2 \left[\zeta_1''(0) + \right. \\
& \left. + \frac{f_1''(0)}{T^2}\right] + t_1 \left[-\frac{5}{4}\zeta_0''(0) - \frac{f_1''(0)}{T}\right] - \frac{1}{3} f_1''(0) = 0
\end{aligned}
$$

Example: A circular cylinder, having the radius $R = 50\,\text{cm}$, is started by an impulse $U_\infty = 10\,\text{cm/s}$, and then, at $T = \frac{3}{2}$ sec, is given an additional constant acceleration $V_0 = 10\,\text{cm/s}^2$. Since for the circular cylinder

$$U = 2\,U_\infty \sin\frac{x}{R}, \quad W = 2\,V_0 \sin\frac{x}{R},$$

for $x = R\pi$, we shall have $\frac{U}{F} = -\frac{5}{4}$, $\frac{W}{F} = -\frac{5}{4}$, and the equation (7.5) becomes

$$
(7.6)\qquad \begin{cases} 0{,}0126\,t_1^8 - 0{,}024\,t_1^7 + 0{,}140\,t_1^6 - 0{,}220\,t_1^5 + \\ + 0{,}344\,t_1^4 - 0{,}546\,t_1^3 = -1{,}807\,t_1^2 + 3{,}570\,t_1 + 0{,}376. \end{cases}
$$

The only real and positive solution of (7.6)

$$t_1 = 1{,}80\,\text{sec}$$

gives the instant of separation of the additional boundary layer.

The expression (7.4) can be used to find the profile of the boundary layer. The shape of that profile may serve as an indication of the procedure introduced and used here.

We shall not attempt any further analysis or more detailed tabular calculations of the present problem (See Ref. (9)). We give only the velocity profiles at several positions along the circular cylinder contour at a specified instant. The shapes of the boundary layer velocities curves (Fig. 9) indicate a natural development of the boundary layer. Obviously, there is some „deterioration“ of the velocitity profile when the zone of the first separation of the boundary layer is approached, and this feature is considered as natural.

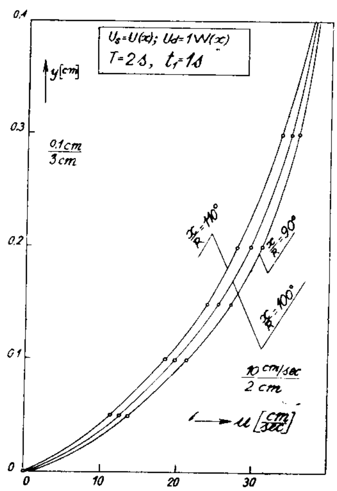

Fig. 9

8. Conclusion. — In addition to these two cases, detailed investigations were carried out on boundary layers on a body started impulsively or with a constant acceleration from the state of a preceding motion with a constant acceleration (9). It was found that the longest duration until separation took place occurs in the case of an additional constantly accelerated motion following a preceding constantly accelerated motion. In this connection, it is interesting to find out whether it is possible to postpone constantly the instant of separation (or to increase the distance of the point of separation) by increasing the exponent „α" of the time series in expressions for the external potential velocities $U_s = At^\alpha W(x)$, $U_d = Bt^\alpha W(x)$. The answer to this question is obtainable only in case of the so-called „short-lived" preceding motions (9). Even for $\alpha \to \infty$, the distance of separation of a circular cylinder, having radius R, remains constant

$$s_{odv} = \frac{1}{\sqrt{2}-1} R$$

This conclusion is in accordance with the well-known result obtained by Watson (10), for a case of motion started from the state of rest.

The basic and most essential advantage of causing additional motions during the course of certain preceding motions is the postponement of the instant of the first separation of the boundary layer, as evidenced by numerical results: $t_1 = 1{,}35$ sec, $t_1 = 1{,}80$ sec.

In fact, this amounts to prolongation of duration of one of the most favourable conditions of velocity within the boundary layer, a condition in which there occurs no separation of the boundary layer. In case the preceding boundary layer separates somewhere on the cylinder itself, the point of separation will be moved further downstream, towards the rear stagnation point, by the introduction of an additional motion (9). This fact is very important in practice. It is impossible, of course, to prolong indefinitely this condition of not having the boundary layer separation by introducing an additional motion. This theory holds good until the infinite velocities become of such order of magnitude that the flow is transformed into a turbulent one.

It is emphasized that these results on postponement of the instant of boundary layer separation in case of motions started from preceding non--steady motions cannot be interpreted as being not in accordance with Blasius' well known formula for the determination of the instant of first separation on a circular cylinder:

$$t_s = 0{,}35 \frac{R}{U_\infty}$$

They cannot be treated in this way because Blasius' result corresponds to motion started directly from the state of rest, while our initial conditions were substantially different. If this were a formal additional motion, but starting at the same time when the preceding one, then we would be justified to demand that the results of investigations for such a case be to some extent in acordance with Blasisus' formula. The equations for that case would essentially differ from our equations (3.2) and (3.4).

In our case, however, the existence of an already partially developed preceding boundary layer at the instant when the additional boundary layer is only to start its formation, represents the central factor which takes its part in the formation of equations (3.2) and (3.4), as clearly indicated by the physical model shown in Fig. 2. This is precisely the feature which differs from the circumstances in which Blasius' formula was obtained, and which precludes its application in our case.

Therefore, the present investigations refer to a qualitatively new and different situation, and thus the results, considered independently of the existing experience, are quite natural. The postponement of the point of separation of the boundary layer, both in space and in time, as a consequence of the introduction of an additional accelerated motion in the same direction as the preceding accelerated motion started from rest — is physically both justified and acceptable.

Thus, in this paper, a solution was found for that most important case of change of the functions u_s, formally given by the relation $0<\eta\leqslant 2$. The remaining interval $2\leqslant\eta\leqslant\infty$, where $u_s=U_s(x, t)$, which is theoretically important mainly because of its vastness, was also considered (9) and will be dealt with in a future paper.

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