PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 6 (20), 1966, p. 65–76

ON A CLASS OF FUNCTIONAL EQUATIONS

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(Communicated June, 25 1965)

§ 1 Introduction

We are concerned with the functional equations

$$\sum_{i=1}^{m+n} f_i(x_i+\cdots+x_{i+m-1}, x_{i+m}+\cdots+x_{i+m+n-1})=0, \tag{1.1}$$

$$\sum_{i=1}^{m+n} f(a^{m-1}x_i+a^{m-2}x_{i+1}+\cdots+ax_{i+m-2}+x_{i+m-1}, a^{n-1}x_{i+m} + a^{n-2}x_{i+m+1}+\cdots+ax_{i+m+n-2}+x_{i+m+n-1})=0, \tag{1.2}$$

where $f_i\,(i=1, 2, \ldots, m+n)$ and f are unknown functions and $x_i\,(i=1, 2, \ldots, m+n)$ are independent variables (the indices must be reduced $\mathrm{mod}\,(m+n)$). The independent variables and the values of all functions are complex.

In §4 we solve a particular case $m=2, n=1$, of the equation

$$\sum_{i=1}^{m+n} f_i(a^{m-1}x_i+a^{m-2}x_{i+1}+\cdots+ax_{i+m-2}+x_{i+m-1}, a^{n-1}x_{i+m} + a^{n-2}x_{i+m+1}+\cdots+ax_{i+m+n-2}+x_{i+m+n-1})=0, \tag{1.3}$$

The equation (1.1) was solved in [3] by one of us under the hypothesis that the functions and variables are real. Let (m, n) be the greatest common divisor of m and n. The theorems of [3] concerning the cases $m\neq n$ should be modified to give the general continuous solutions. In the more general formulation as given in [3] they are invalid.

§ 2

Let C be the field of complex numbers and $f_i: C^2\to C\,(i=1, 2, \ldots, m+n)$ unknown functions such that

$$\sum_{i=1}^{m+n} f_i(x_i+\cdots+x_{i+m-1}, x_{i+m}+\cdots+x_{i+m+n-1})=0 \tag{2.1}$$

where $x_i\in C\,(i=1, 2, \ldots, m+n)$ are independent variables.

Theorem 1. *The general continuous solution of the functional equation* (2.1) *in the case* $(m,n)=1$, $m+n>2$ *is*

$$f_i(x,y)=F_1(x+y)\,Re(x)+F_2(x+y)\,Im(x)+G_i(x+y)\ (i=1,2,\ldots,m+n), \tag{2.2}$$

$$\sum_{i=1}^{m+n} G_i(x)=-m\,[F_1(x)\,Re(x)+F_2(x)\,Im(x)],$$

where $F_1, F_2, G_i\colon C\to C\ (i=1,2,\ldots,m+n-1)$ *are arbitrary continuous functions.*

Proof. If we set (the indices have to be reduced $\mathrm{mod}(m+n)$)

$$s=x_1+x_2+\cdots+x_{m+n}$$

$$t_i=x_i+x_{i+1}+\cdots+x_{i+m-1}-\frac{ms}{m+n}\quad(i=1,2,\ldots,m+n-1) \tag{2.3}$$

the variables $t_i\ (i=1,2,\ldots,m+n-1)$ and s are independent since $(m,n)=1$ (see [3]). The equation (2.1) becomes

$$\sum_{i=1}^{m+n-1} f_i\left(t_i+\frac{ms}{m+n},\ \frac{ns}{m+n}-t_i\right) \tag{2.4}$$

$$+f_{m+n}\left(-t_1-t_2-\cdots-t_{m+n-1}+\frac{ms}{m+n},\ \frac{ns}{m+n}+t_1+t_2+\cdots+t_{m+n-1}\right)=0.$$

We introduce the new notations

$$f_i\left(x+\frac{ms}{m+n},\ \frac{ns}{m+n}-x\right)=g_i(x,\ s)\quad(i=1,2,\ldots,m+n),$$

i.e.

$$f_i(x,y)=g_i\left(\frac{nx-my}{m+n},\ x+y\right)\quad(i=1,2,\ldots,m+n). \tag{2.5}$$

The equation (2.4) is transformed into

$$\sum_{i=1}^{m+n-1} g_i(t_i,s)+g_{m+n}(-t_1-t_2-\cdots-t_{m+n-1},\ s)=0. \tag{2.6}$$

By substitution $t_1=t_2=\cdots=t_{r-1}=t_{r+1}=\cdots=t_{m+n-1}=0$, we arrive at

$$g_r(t_r,s)=-g_{m+n}(-t_r,\ s)-H_r(s)\quad(r=1,2,\ldots,m+n-1). \tag{2.7}$$

Putting (2.7) into (2.6) we get

$$g_{m+n}(-t_1-t_2-\cdots-t_{m+n-1},s)=\sum_{i=1}^{m+n-1} g_{m+n}(-t_i,s)+\sum_{i=1}^{m+n-1} H_i(s). \tag{2.8}$$

We conclude that the function

$$K(x,s)=g_{m+n}(x,s)+\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(s) \tag{2.9}$$

satisfies the functional equation

$$K(x_1+x_2+\cdots+x_{m+n-1},s)=\sum_{i=1}^{m+n-1} K(x_i,\ s), \tag{2.10}$$

From (2.10) using continuity of K, we deduce that for fixed s (see [1])

$$K(x,s)=c_1\,Re\,(x)+c_2\,Im\,(x),$$

where $Re\,(x)$ resp. $Im\,(x)$ denotes the real resp. imaginary part of x. The constants c_1, c_2 may depend upon s. Hence,

$$K(x,y)=F_1(y)\,Re\,(x)+F_2(y)\,Im\,(x), \tag{2.11}$$

where $F_1, F_2\colon C\to C$ are continuous.

From (2.9), (2.11) and (2.7) we obtain

$$\begin{aligned} g_{m+n}(x,y)&=F_1(y)\,Re\,(x)+F_2(y)\,Im\,(x)-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1}H_i(y),\\ g_r(x,y)&=F_1(y)Re(x)+F_2(y)\,Im\,(x)-H_r(y)+\frac{1}{m+n-2}\sum_{i=1}^{m+n-1}H_i(y)\\ &(r=1,2,\dots,m+n-1). \end{aligned} \tag{2.12}$$

From (2.5) and (2.12) we deduce that

$$\begin{aligned} f_r(x,y)&=F_1(x+y)\,Re\left(\frac{nx-my}{m+n}\right)+F_2(x+y)\,Im\left(\frac{nx-my}{m+n}\right)-H_r(x+y)\\ &+\frac{1}{m+n-2}\sum_{i=1}^{m+n-1}H_i(x+y) \qquad (r=1,2,\dots,m+n-1),\\ f_{m+n}(x+y)&=F_1(x+y)\,Re\left(\frac{nx-my}{m+n}\right)+F_2(x+y)\,Im\left(\frac{nx-my}{m+n}\right)\\ &-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1}H_i(x+y). \end{aligned} \tag{2.13}$$

By denoting

$$-F_1(x+y)\,Re\left(\frac{m(x+y)}{m+n}\right)-F_2(x+y)\,Im\left(\frac{m(x+y)}{m+n}\right)+\frac{1}{m+n-2}\sum_{i=1}^{m+n-1}H_i(x+y)$$

$$-H_r(x+y)=G_r(x+y) \qquad (r=1,2,\dots,m+n-1);$$

$$-F_1(x+y)\,Re\left(\frac{m(x+y)}{m+n}\right)-F_2(x+y)\,Im\left(\frac{m(x+y)}{m+n}\right)$$

$$-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1}H_i(x+y)=G_{m+n}(x+y)$$

from (2.13) we get (2.2).

The converse can be established by direct verification.

Example. The general continuous solution of functional equation

$$f_1(x+y,z)+f_2(y+z,x)+f_3(z+x,y)=0,$$

is given by

$$f_1(x,y)=F_1(x+y)\,Re\,(x)+F_1(x+y)\,Im\,(x)+G_1(x+y),$$

$$f_2(x,y)=F_1(x+y)\,Re\,(x)+F_2(x+y)\,Im\,(x)+G_2(x+y),$$

$$f_3(x,y)=-F_1(x+y)\,Re\,(x+2y)-F_2(x+y)\,Im\,(x+2y)-G_1(x+y)-G_2(x+y),$$

where F_1, F_2, G_1, G_2 are arbitrary continuous functions.

Corollary. *The general continuous solution of the functional equation*

$$\sum_{i=1}^{m+n} g_i(x_i + \cdots + x_{i+m-1}, x_1 + x_2 + \cdots + x_{m+n}) = 0$$

if $(m, n) = 1$, $m+n > 2$, *is given by*

$$g_i(x, y) = F_1(y)\, Re\,(x) + F_2(y)\, Im\,(x) + G_i(y) \quad (i = 1, 2, \ldots, m+n),$$

$$\sum_{i=1}^{m+n} G_i(y) = -m[F_1(y)\, Re\,(y) + F_2(y)\, Im\,(y)],$$

where $F_1, F_2, G_i \colon C \to C$ $(i = 1, 2, \ldots, m+n-1)$ *are arbitrary continuous functions.*

Proof. Put $f_i(x, y) = g_i(x, x+y)$ in Theorem 1.

Theorem 2. *The general continuous solution of the functional equation* (2.1) *in the case* $(m, n) = d > 1$, $m/d = \mu$, $n/d = \nu$, $\mu + \nu > 2$, *is given by*

$$f_{id+j}(x, y) = F_1^j(x+y)\, Re\,(x) + F_2^j(x+y)\, Im\,(x) + G_i^j(x+y)$$
$$(i = 0, 1, \ldots, \mu+\nu-1;\ j = 1, 2, \ldots, d),$$

(2.14) $$\sum_{i=0}^{\mu+\nu-1} G_i^j(x) = H^j(x) - \mu\,[F_1^j(x)\, Re\,(x) + F_2^j(x)\, Im\,(x)] \quad (j = 1, 2, \ldots, d),$$

$$\sum_{j=1}^{d} H^j(x) = 0,$$

where F_1^j, F_2^j $(j = 1, 2, \ldots, d)$, H^j $(j = 1, 2, \ldots, d-1)$, G_i^j $(i = 0, 1, \ldots, \mu+\nu-2;$ $j = 1, 2, \ldots, d)$ *are arbitrary continuous complex-valued functions of a complex variable.*

Proof. We set

(2.15) $$f_i(x, y) = g_i(x, x+y) \quad (i = 1, 2, \ldots, m+n)$$

and we obtain

(2.16) $$\sum_{i=1}^{m+n} g_i(x_i + x_{i+1} + \cdots + x_{i+m-1}, x_1 + x_2 + \cdots + x_{m+n}) = 0.$$

Let us introduce the new variables

(2.17) $$y_i = x_i + x_{i+1} + \cdots + x_{i+d-1} \quad (i = 1, 2, \ldots, m+n), \quad y_{i+m+n} = y_i,$$

and

(2.18) $$z = x_1 + x_2 + \cdots + x_{m+n}.$$

They are not independent since

(2.19) $$\sum_{i=0}^{\mu+\nu-1} y_{id+j} = z \quad (j = 1, 2, \ldots, d).$$

The variables y_i $(i = 1, 2, \ldots, m+n-d)$, and z are independent since it is easy to see that the rank of the matrix of linear forms determining them is

$m+n-d+1$. In the sequel we shall use all variables (2.17) and (2.18) but we must have always in mind that (2.19) holds. The equation (2.16) is now

$$\sum_{i=1}^{m+n} g_i(y_i+y_{i+d}+\cdots+y_{i+(\mu-1)d}, z)=0.$$

It can be rewritten in the form

$$\sum_{j=1}^{d}\sum_{i=0}^{\mu+\nu-1} g_{id+j}(y_{id+j}+y_{(i+1)d+j}+\cdots+y_{(i+\mu-1)d+j}, z)=0.$$

If we set here

$$y_{id+j}=0 \quad (i=0,1,\ldots,\mu+\nu-2;\ j=1,2,\ldots,r-1,r+1,\ldots,d),$$
$$y_{(\mu+\nu-1)d+j}=z \quad (j=1,2,\ldots,r-1,r+1,\ldots,d),$$

we get

$$\sum_{i=0}^{\mu+\nu-1} g_{id+r}(y_{id+r}+y_{(i+1)d+r}+\cdots+y_{(i+\mu-1)d+r}, z)-\frac{H^r(z)}{\mu+\nu}=0$$
$$(r=1,2,\ldots,d),$$

and

$$\sum_{r=1}^{d} H^r(z)=0.$$

On using Corollary of Theorem 1 we get

$$g_{id+r}(x,y)=F_1^r(y)\,Re(x)+F_2^r(y)\,Im(x)+G_i^r(y)$$
$$(i=0,1,2,\ldots,\mu+\nu-1;\ r=1,2,\ldots,d)$$

$$\sum_{i=0}^{\mu+\nu-1} G_i^r(y)=H^r(y)-\mu\,[F_1^r(y)\,Re(y)+F_2^r(y)\,Im(y)] \quad (r=1,2,\ldots,d),$$

where F_1^r, F_2^r $(r=1,2,\ldots,d)$, G_i^r $(i=0,1,\ldots,\mu+\nu-1;\ r=1,2,\ldots,d)$, H^r $(r=1,2,\ldots,d)$ are continuous functions. By application of (2.15) these formulae give (2.14).

It is easy to verify that the functions $f_i\colon C\to C$ $(i=1,2,\ldots,m+n)$ defined by (2.15) satisfy the functional equation (2.1).

The theorem is proved.

Example. The general continuous solution of the functional equation

$$f_1(x+y+z+u, v+w)+f_2(y+z+u+v, w+x)+f_3(z+u+v+w, x+y)$$
$$+f_4(u+v+w+x, y+z)+f_5(v+w+x+y, z+u)+f_6(w+x+y+z, u+v)=0,$$

is given by

$$f_1(x,y)=F_1^1(x+y)\,Re(x)+F_2^1(x+y)\,Im(x)+G_0^1(x+y),$$

$$f_2(x,y)=F_1^2(x+y)\,Re(x)+F_2^2(x+y)\,Im(x)+G_0^2(x+y),$$

$$f_3(x,y)=F_1^1(x+y)\,Re(x)+F_2^1(x+y)\,Im(x)+G_1^1(x+y),$$

$$f_4(x,y)=F_1^2(x+y)\,Re(x)+F_2^2(x+y)\,Im(x)+G_1^2(x+y),$$

$$f_5(x, y) = -F_1^1(x+y)\, Re(x+2y) - F_2^1(x+y)\, Im(x+2y) + H^1(x+y) - G_0^1(x+y) - G_1^1(x+y),$$

$$f_6(x, y) = -F_1^2(x+y)\, Re(x+2y) - F_2^2(x+y)\, Im(x+2y) - H^1(x+y) - G_0^2(x+y) - G_1^2(x+y),$$

where F_i^j, G_i^j, H^1 are arbitrary continuous functions.

Theorem 3. *The most general solution of* (2.1) *if* $m=n$ *is*

$$f_i(x, y) \quad (i=1, \dots, m) \text{ are arbitrary;}$$

$$f_{m+i}(x, y) = H_i(x+y) - f_i(y, x) \quad (i=1, \dots, m), \tag{2.20}$$

$$\sum_{i=1}^{m} H_i(x) = 0,$$

where $H_i: C \to C$ $(i=1, 2, \dots, m-1)$ *are arbitrary functions.*

The proof given in [3] for the case of real functions remains valid also for complex-valued functions of complex variables.

Example. The most general solution of

$$f_1(x+y, z+u) + f_2(y+z, u+x) + f_3(z+u, x+y) + f_4(u+x, y+z) = 0,$$

is

$$f_1(x, y),\ f_2(x, y) \text{ are arbitrary,}$$

$$f_3(x, y) = H_1(x+y) - f_1(y, x),$$

$$f_4(x, y) = -H_1(x+y) - f_2(y, x),$$

where H_1 is arbitrary function.

§ 3

Now we consider the functional equation

$$\sum_{i=0}^{m+n} f(a^{m-1} x_i + a^{m-2} x_{i+1} + \cdots + a x_{i+m-2} + x_{i+m-1},\ a^{n-1} x_{i+m} + a^{n-2} x_{i+m+1} + \cdots + a x_{i+m+n-2} + x_{i+m+n-1}) = 0, \tag{3.1}$$

where $a \in C$ is a constant, f: $C \to C$ is an unknown function and $x_1, x_2, \dots, x_{m+n}$ are independent complex variables. The indices have to be reduced mod $m+n$.

Theorem 4. *If* $a^{m+n} \neq 1$, *the most general solution of the functional equation* (3.1) *is given by*

$$\begin{aligned} f(x, y) &= F(x + a^m y) - F(a^n x + y) \quad (m \neq n), \\ f(x, y) &= S(x + a^m y, a^m x + y) - S(a^m x + y, x + a^m y) \quad (m=n), \end{aligned} \tag{3.2}$$

where $F: C \to C$, $S: C^2 \to C$ *are arbitrary functions.*

Proof. We set

$$f(x, y) = g(x + a^m y, a^n x + y) \tag{3.3}$$

in (3.1) and deduce that

$$\sum_{i=0}^{m+n-1} g\left(\sum_{k=0}^{m+n-1} a^k x_{m+i-k},\ \sum_{k=0}^{m+n-1} a^k x_{i-k}\right) = 0. \tag{3.4}$$

We notice that the linear forms $x + a^m y$ and $a^m x + y$ are independent since $a^{m+n} \neq 1$.

Now we introduce the new variables

$$(3.5)\qquad y_i=\sum_{k=0}^{m+n-1} a^k x_{i-k} \quad (i=0, 1, \ldots, m+n-1).$$

The linear forms (3.5) are independent since their determinant is $(a^{m+n}-1)^{m+n-1}$.

Making use of these notations the equation (3.4) becomes

$$(3.6)\qquad \sum_{i=1}^{m+n} g(y_i, y_{i+n})=0.$$

If $m\neq n$ we set $y_1=y_2=\cdots=y_{m-1}=y_{m+1}=y_{m+2}=\cdots=y_{m+n-1}=0$ and we get

$$(3.7)\qquad g(x,y)=F(x)+G(y).$$

We substitute g from (3.7) into (3.6) and obtain

$$\sum_{i=1}^{m+n}(F(y_i)+G(y_i))=0$$

where from it follows that $F(y)=-G(y)$. Hence,

$$(3.8)\qquad g(x,y)=F(x)-F(y).$$

If $m=n$ the equation (3.6) gives $g(x,y)+g(y,x)=0$, i.e.

$$(3.9)\qquad g(x,y)=S(x,y)-S(y,x).$$

From (3.3), (3.8) and (3.9) we conclude that (3.2) holds. It is easy to verify that (3.2) satisfies (3.1).

The Theorem 4 is proved.

Example 1. If $a^3\neq 1$ the most general solution of the functional equation

$$f(ax+y, z)+f(ay+z, x)+f(az+x, y)=0$$

is given by

$$f(x, y)=F(x+a^2 y)-F(ax+y),$$

where F is arbitrary.

Example 2. If $a^4\neq 1$ the most general solution of the functional equation

$$f(ax+y, az+u)+f(ay+z, au+x)+f(az+u, ax+y)+f(au+x, ay+z)=0$$

is given by

$$f(x, y)=S(x+a^2 y, a^2 x+y)-S(a^2 x+y, x+a^2 y),$$

where S is arbitrary.

Theorem 5. *If* $a^{m+n}=1$, $(m,n)=1$, $m+n>2$, *the general continuous solution of the functional equation* (3.1) *is given by*

$$(3.10)\qquad f(x,y)=\sum_{i=1}^{m+n}[F_1(a^i x+a^{i+m} y)\,Re\,(a^i x)+F_2(a^i x+a^{i+m} y)\,Im\,(a^i x)]$$

$$+\sum_{i=1}^{m+n-1}[G_i(a^i x+a^{i+m} y)-G_i(x+a^m y)]$$

$$-m[F_1(x+a^m y)\,Re\,(x+a^m y)+F_2(x+a^m y)\,Im\,(x+a^m y)],$$

where $F_1, F_2, G_i\colon C \to C$ $(i = 1, 2, \ldots, m+n-1)$ *are arbitrary continuous functions.*

Proof. Let us put $x_i = a^{i-1} X_i$ $(i = 1, 2, \ldots, m+n)$. The equation (3.1) becomes

$$\sum_{i=1}^{m+n} f\left(a^{m+i-2} \sum_{k=i}^{m+i-1} X_k,\ a^{i-2} \sum_{k=m+i}^{m+n+i-1} X_k\right) = 0. \tag{3.11}$$

Now we make the substitutions

$$f(a^{m+i-2} x, a^{i-2} y) = f_i(x, y) \quad (i = 1, 2, \ldots, m+n),$$

i.e.

$$f(x, y) = f_i(a^{n-i+2} x, a^{2-i} y) \quad (i = 1, 2, \ldots, m+n), \tag{3.12}$$

and we obtain

$$\sum_{i=1}^{m+n} f_i(X_i + X_{i+1} + \cdots + X_{i+m-1}, X_{i+m} + X_{i+m+1} + \cdots + X_{i+m+n-1}) = 0. \tag{3.13}$$

By application of Theorem 1, and by (3.12), we get

$$\begin{aligned} f_i(x, y) = {} & A_1(a^{n-i+2} x + a^{2-i} y)\, Re\,(a^{n-i+2} x) \\ & + A_2(a^{n-i+2} x + a^{2-i} y)\, Im\,(a^{n-i+2} x) + B_i(a^{n-i+2} x + a^{2-i} y) \\ & (i = 1, 2, \ldots, m+n), \end{aligned} \tag{3.14}$$

$$\sum_{i=1}^{m+n} B_i(x) = -m\,[A_1(x)\, Re\,(x) + A_2(x)\, Im\,(x)],$$

where $A_1, A_2, B_i\colon C \to C$ $(i = 1, 2, \ldots, m+n)$ are continuous functions. By addition of all equations (3.14) and putting

$$A_1(x) = (m+n) F_1(x), \quad A_2(x) = (m+n) F_2(x), \quad B_i(x) = (m+n) G_{n+2-i}(x)$$
$$(i = 1, 2, \ldots, n+1, n+3, \ldots, m+n)$$

we obtain (3.10).

Example. If $a^3 = 1$ the most general continuous solution of the functional equation

$$f(ax+y, z) + f(ay+z, x) + f(az+x, y) = 0$$

is given by

$$\begin{aligned} f(x, y) = {} & F_1(ax+y)\, Re\,(ax) + F_2(ax+y)\, Im\,(ax) \\ & + F_1(a^2 x + ay)\, Re\,(a^2 x) + F_2(a^2 x + ay)\, Im\,(a^2 x) \\ & - F_1(x + a^2 y)\, Re\,(x + 2a^2 y) - F_2(x + a^2 y)\, Im\,(x + 2a^2 y) \\ & + G_1(ax+y) - G_1(x + a^2 y) \\ & + G_2(a^2 x + ay) - G_2(x + a^2 y), \end{aligned}$$

where F_1, F_2, G_1, G_2 are arbitrary continuous functions.

For $a = 1$ we get a theorem of [2].

Theorem 6. *If* $a^{m+n} = 1$ *and* $(m, n) = d > 1$, $m/d = \mu$, $n/d = \nu$, $\mu + \nu > 2$, *the general continuous solution of the functional equation* (3.1) *is given by*

$$\begin{aligned} f(x, y) = \sum_{j=-1}^{d-2} \sum_{i=0}^{\mu+\nu-1} & [F_1^{j+2}(a^{n-id-j} x + a^{-id-j} y)\, Re\,(a^{n-id-j} x) \\ & + F_2^{j+2}(a^{n-id-j} x + a^{-id-j} y)\, Im\,(a^{n-id-j} x) + G_i^{j+2}(a^{n-id-j} x + a^{-id-j} y)], \end{aligned} \tag{3.15}$$

$$\sum_{i=0}^{\mu+\nu-1} G_i^j(x) = H^j(x) - \mu\,[F_1^j(x)\,Re\,(x) + F_2^j(x)\,Im\,(x)] \qquad (j=1,2,\dots,d),$$

$$\sum_{j=1}^{d} H^j(x) = 0,$$

where F_1^j, F_2^j, G_i^j $(i=0,1,\dots,\mu+\nu-2;\ j=1,2,\dots,d)$, H^j $(j=1,2,\dots,d-1$ *are arbitrary continuous complex-valued functions.*

Proof. We can start from equation (3.13). From (3.12) and (3.13) on the basis of Theorem 2 we get

$$f(x,y) = A_1^j(a^{n-id-j+2}x + a^{2-id-j}y)\,Re\,(a^{n-id-j+2}x) \tag{3.16}$$
$$+ A_2^j(a^{n-id-j+2}x + a^{2-id-j}y)\,Im\,(a^{n-id-j+2}x) + B_i^j(a^{n-id-j+2}x + a^{2-id-j}y)$$
$$(i=0,1,\dots,\mu+\nu-1;\ j=1,2,\dots,d),$$

$$\sum_{i=0}^{\mu+\nu-1} B_i^j(x) = C^j(x) - \mu[A_1^j(x)\,Re(x) + A_2^j(x)\,Im\,(x)] \qquad (j=1,2,\dots,d), \tag{3.17}$$

$$\sum_{j=1}^{d} C^j(x) = 0 \tag{3.18}$$

where A_1^j, A_2^j, B_i^j, $C^j\colon C \to C$ are continuous functions.

We take into account (3.17) and (3.18) and we add all equations (3.16). In that way we obtain (3.15) with

$$A_1^j(x) = (m+n)\,F_1^j\,(x), \quad A_2^j(x) = (m+n)\,F_2^j(x),$$
$$B_i^j(x) = (m+n)\,G_i^j(x), \quad C^j(x) = (m+n)\,H^j(x)$$
$$(i=0,1,\dots,\mu+\nu-2;\ j=1,2,\dots,d).$$

Example. If $a^6=1$ the most general continuous solution of the functional equation

$$f(a^3x+a^2y+az+u,\ av+w) + f(a^3y+a^2z+au+v,\ aw+x) + f(a^3z+a^2u+av+w,\ ax+y)$$
$$+ f(a^3u+a^2v+aw+x,\ ay+z) + f(a^3v+a^2w+ax+y,\ az+u) + f(a^3w+a^2x+ay+z,\ au+v) = 0$$

is given by

$$f(x,y) = F_1^1(ax+a^5y)\,Re\,(ax) + F_2^1(ax+a^5y)\,Im\,(ax)$$
$$+ F_1^1(a^3x+ay)\,Re\,(a^3x) + F_2^1(a^3x+ay)\,Im\,(a^3x)$$
$$- F_1^1(a^5x+a^3y)\,Re\,(a^5x+2a^3y) - F_2^1(a^5x+a^3y)\,Im\,(a^5x+2a^3y)$$
$$+ F_1^2(x+a^4y)\,Re\,(x) + F_2^2(x+a^4y)\,Im\,(x)$$
$$+ F_1^2(a^2x+y)\,Re\,(a^2x) + F_2^2(a^2x+y)\,Im\,(a^2x)$$
$$- F_1^2(a^4x+a^2y)\,Re\,(a^4x+2a^2y) - F_2^2(a^4x+a^2y)\,Im\,(a^4x+2a^2y)$$
$$+ G_0^1(ax+a^5y) - G_0^1(a^5x+a^3y) + G_0^2(x+a^4y) - G_0^2(a^4x+a^2y)$$
$$+ G_1^1(a^3x+ay) - G_1^1(a^5x+a^3y) + G_1^2(a^2x+y) - G_1^2(a^4x+a^2y)$$
$$+ H^1(a^5x+a^3y) - H^1(a^4x+a^2y),$$

where F_1^1, F_1^2, F_2^1, F_2^2, G_0^1, G_0^2, G_1^1, G_1^2, H^1 are arbitrary continuous functions.

Theorem 7. *If* $a^{m+n}=1$, $m=n$, *the most general solution of the functional equation* (3.1) *is given by*

$$f(x,y)=\sum_{i=1}^{m}[F_i(a^i x, a^{n+i} y)-F_i(a^i y, a^{n+i} x)+H_i(a^{n+i}x+a^i y)],$$

(3.19)

$$\sum_{i=1}^{m} H_i(x)=0,$$

where F_i $(i=1,2,\ldots,m)$, H_i $(i=1,2,\ldots,m-1)$ *are arbitrary complex-valued functions.*

Proof. We start again from the equation (3.13). According to Theorem 3 and (3.12) we have

$$f(x,y)=A_i(a^{m-i+2}x, a^{2-i}y) \quad (i=1,2,\ldots,m)$$

$$f(x,y)=B_i(a^{m-i+2}x+a^{2-i}y)-A_i(a^{2-i}y, a^{m-i+2}x) \quad (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} B_i(x)=0.$$

By addition we get (3.19) with

$$A_i(x,y)=2\,m\,F_{m-i+2}(x,y),\; B_i(x)=2\,m\,H_{m-i+2}(x).$$

Example. If $a^4=1$, the most general solution of the functional equation

$$f(ax+y, az+u)+f(ay+z, au+x)+f(az+u, ax+y)+f(au+x, ay+z)=0$$

is given by

$$f(x,y)=F_1(ax, a^3y)-F_1(ay, a^3x)+F_2(a^2x, y)-F_2(a^2y, x)+H_1(a^3x+ay)-H_1(x+a^2y),$$

where F_1, F_2, H_1, are arbitrary functions.

§ 4

Let $f_i\colon C^2\to C\,(i=1,2,\ldots,m+n)$ be unknown functions such that

(4.1) $$f_1(ax+y,z)+f_2(ay+z,x)+f_3(az+x,y)=0,$$

where x,y,z are independent complex variables and $a\in C$ is a constant.

Theorem 8. *If* $a^3\neq 1$ *the most general solution of the functional equation* (4.1) *is*

(4.2) $$f_1(x,y)=F_2(ax+a^3y)-F_2(ax+y)+F_3(x+a^2y),$$

(4.3) $$f_2(x,y)=F_1(ax+y)+F_2(x+a^2y),$$

(4.4) $$f_3(x,y)=-F_1(x+a^2y)-F_2(a^2x+ay)-F_3(ax+y),$$

where $F_i\colon C\to C$ $(i=1,2,3)$ *are arbitrary.*

Proof. By setting $z=0$ in (4.1) we obtain

(4.5) $$f_3(x,y)=-f_1(ax+y,0)-f_2(ay,x).$$

Putting back (4.5) into (4.1) we get

(4.6) $$f_1(ax+y,z)+f_2(ay+z,x)-f_2(ay,az+x)-f_1(a^2z+ax+y,0)=0.$$

For $x=0$ we arrive at

$$f_1(y,z)+f_2(ay+z,0)-f_2(ay,az)-f_1(a^2z+y,0)=0. \tag{4.7}$$

If we replace y by $ax+y$ we deduce from (4.7) that

$$f_1(ax+y,z)+f_2(a^2x+ay+z,0)-f_2(a^2x+ay,az)-f_1(a^2z+ax+y,0)=0.$$

The subtraction of the last equality from (4.6) gives

$$f_2(ay+z,x)+f_2(a^2x+ay,az)-f_2(ay,az+x)-f_2(a^2x+ay+z,0)=0. \tag{4.8}$$

We can set

$$f_2(x,y)=F(ax+y,x+a^2y). \tag{4.9}$$

We notice that the linear forms $ax+y$ and $x+a^2y$ are independent since $a^3\neq 1$. The equation (4.8) becomes

$$F(a^2y+az+x,a^2x+ay+z)+F(a^3x+a^2y+az,a^3z+a^2x+ay) \tag{4.10}$$
$$-F(a^2y+az+x,a^3z+a^2x+ay)-F(a^3x+a^2y+az,a^2x+ay+z)=0.$$

If $a\neq 0$ the new variables $u=a^2y+az+x$, $v=a^2x+ay+z$, $w=a^3z+a^2x+$ $+ay$, are independent. The determinant of these forms is $-a(a^3-1)^2\neq 0$. Making use of u,v,w, the equation (4.10) becomes

$$F(u,v)+F(av,w)-F(u,w)-F(av,v)=0. \tag{4.11}$$

Setting $w=0$, we get

$$F(u,v)=F_1(u)+F_2(v). \tag{4.12}$$

In the case $a=0$, (4.12) follows immediately from (4.10). The formula (4.3) follows from (4.9) and (4.12). Using (4.3) and (4.7) and putting $f_1(x,0)=$ $=F_3(x)$ we obtain (4.2). The equation (4.4) follows from (4.2), (4.3) and (4.5).

We have shown that every solution of (4.1) if $a^3\neq 1$ can be written in the form (4.2), (4.3), (4.4). The converse can be easily verified.

Theorem 9. *If* $a^3=1$ *the general continuous solution of the functional equation* (4.1) *is*

$$\begin{aligned} f_1(x,y)&=F_1(a^2x+ay)\,Re(a^2x)+F_2(a^2x+ay)\,Im(a^2x)+G_1(a^2x+ay),\\ f_2(x,y)&=F_1(ax+y)\,Re(ax)+F_2(ax+y)\,Im(ax)+G_2(ax+y),\\ f_3(x,y)&=-F_1(x+a^2y)\,Re(x+2a^2y)-F_2(x+a^2y)\,Im(x+2a^2y)\\ &\quad -G_1(x+a^2y)-G_2(x+a^2y),\end{aligned} \tag{4.13}$$

where F_1, F_2, G_1, G_2: $C\to C$ *are arbitrary continuous functions.*

Proof. If we set $x=X$, $y=aY$, $z=a^2Z$ in (4.1) we get

$$f_1(a(X+Y),a^2Z)+f_2(a^2(Y+Z),X)+f_3(Z+X,aY)=0, \tag{4.14}$$

since $a^3=1$. Let us put

$$\begin{aligned} f_1(ax,a^2y)&=g_1(x,y) \quad \text{i. e.} \quad f_1(x,y)=g_1(a^2x,ay),\\ f_2(a^2x,y)&=g_2(x,y) \quad \text{i. e.} \quad f_2(x,y)=g_2(ax,y),\\ f_3(x,ay)&=g_3(x,y) \quad \text{i. e.} \quad f_3(x,y)=g_3(x,a^2y).\end{aligned} \tag{4.15}$$

The equation (4.14) gives

$$g_1(X+Y, Z)+g_2(Y+Z, X)+g_3(Z+X, Y)=0.$$

On applying Theorem 1 we find

$$
\begin{aligned}
&g_1(x,y)=F_1(x+y)\,Re(x)+F_2(x+y)\,Im(x)+G_1(x+y),\\
(4.16)\qquad &g_2(x,y)=F_1(x+y)\,Re(x)+F_2(x+y)\,Im(x)+G_2(x+y),\\
&g_3(x,y)=F_1(x+y)\,Re(x)+F_2(x+y)\,Im(x)-G_1(x+y)-G_2(x+y)\\
&\qquad -2\,F_1(x+y)\,Re(x+y)-2\,F_2(x+y)\,Im(x+y),
\end{aligned}
$$

where F_1, F_2, G_1, G_2: $C\to C$ are continuous functions. Formulae (4.13) follow from (4.15) and (4.16).

Hence, every system of continuous functions f_1, f_2, f_3: $C^2\to C$ which satisfy (4.1) in the case $a^3=1$ has the form (4.13). The converse is also true which can be easily verified. This completes the proof of Theorem 9.

Note. During the print of this article the authors have got the most general solution of the equation (1.3) in the case where m and n are arbitrary. This result will be published in the periodical *Matematički vesnik.*

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