PUBLICATIONS DE L'INSTITUT MATHEMATIQUE
Nouvelle, série tome 6 (20), 1966. p. 29—55

A METHOD FOR SOLUTION OF UNSTEADY INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS

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(Communicated June 29, 1965)

In this paper is given a method for solution of problems of unsteady incompressible laminar boundary-layers, where it is assumed that the velocity of the external potential flow is given in such a form to separate the variables, namely $U(x, t) = V(x) \cdot \Omega(t)$. Principally, the practical problems will be given in just this form, either that they are obtained by way of theoretical, or by experimental consideration of the external potential flow.

1. Introduction

There are not many methods for solution of unsteady boundary layers in comparison to the steady ones. Namely, there are only methods based on the momentum-integral equations, as there are [1], [6], [8] and others, while there are not general methods of Görtler's type [3] from the class of solutions based on exact equations of motion. But, there have been solved a set of special problems, so Watson [10] has solved the case of degree and exponential change of velocity on time; Yang [12] has analysed the stagnation point flow; Moore [5] has considered the case, in which the dependence of velocity at the flow of the plate on time is an arbitrary function. Furthermore, Schlichting [7], Sexl [9] and many others have solved the periodic boundary layers. Hassan [4] has recently done a solution for a certain class of problems on unsteady laminar boundary layers by introducing transformations which reduce the existing equations to equations in which the time does not appear explicitly. Here, we have only shown some of sets of solved cases so far.

In the present paper as well as in [3], we wanted the reduction to similar solutions to be the essence, respectively, that in the process of forming solutions the first term which would satisfy the boundary conditions, would be a *similar solution,* while other terms in this process should only bring the correction of this one within the boundary layer along the contour and on time. To attain this, it was necessary to introduce new variables instead of old ones, which would in themselves contain data for each particular problem which was under consideration. If we leave coordinates x unvariable, and instead of variables y and t introduce the modified stream and potential function of the potential flow around the contour which allows similar solutions, then we shall satisfy the given postulate. In the paper [2] was shown that the plate allows similar solutions.

If the velocity of the potential flow on the plate is $\Omega(t)$, then from the Cauchy-Langrange equation we obtain the expression for the potential function

$$\varphi = \int_0^t \Omega^2(t)\,dt, \tag{1.1}$$

while for the stream function, it can be easily obtained the expression

$$\varphi = \Omega(t)\,y. \tag{1.2}$$

With these magnitudes we shall formulate the transitory coordinates

$$x, \quad \eta^* = \psi, \quad \tau^* = \varphi. \tag{1.3}$$

But to give answer to the above given postulate, respectively that in the case of similar solutions, our transitory variable η^* should be reduced to the unique variable of similar solutions, it was necessary to modify our transitory variables (1.3). Therefore, the ultimately new independent variables for solution of problems of unsteady boundary layers around the contour of the arbitrary form will be

$$x, \quad \eta = \frac{1}{\nu\sqrt{3\tau}}\eta^*, \qquad \tau = \frac{1}{\nu}\tau^*, \tag{1.4'}$$

respectively

$$x, \quad \eta = \frac{\Omega y}{\nu\sqrt{3\tau}}, \qquad \tau = \frac{1}{\nu}\int_0^t \Omega^2\,dt. \tag{1.4}$$

The solution then will be given in the form of special series by the function $V(x)$ and its derivatives with coefficients which are functions of independent variables η and τ. For determination of coefficients of this series we use the obtained system of partial equations. The solution of this system, respectively the coefficients of special series are given in terms of power series by τ, with coefficients which are functions of the reduced distance η from the wall. These coefficients — functions will be given in term of linear combinations of universal functions, which can be tabulated once for ever.

In this paper is given also the classification of problems which can be solved by this method, and is given the method for its application. The application of the method is very simple. Namely, in the domain of convergence, the calculation would be simply effectuated by the time t (respectively τ) i.e. the coefficients-functions of special series could be found, and then would conduct us along the x—axe, and would give us magnitudes of the boundary layer on the single points on the contour.

2. The transformation of basic equations

The differential equations of plane and unsteady incompressible laminar boundary layers are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.2}$$

where: x — distance along the wall of the contour which is flown round by fluid; y — the normal distance from the wall; t — the time; $u(x,y,t)$ and $v(x,y,t)$ — are the components of velocity in the direction of x, respectively y-axis, ν-the constant kinematic viscosity and $U(x,t)=V(x)\cdot\Omega(t)$ — in advance given velocity of the external potential current on the boundary of the boundary layer. The boundary conditions are

$$(2.3)\qquad \begin{aligned} &u=U(x,t),\ v=0, && y=0 \text{ and } t=0,\\ &u=v=0, && y=0 \text{ and } t>0,\\ &u=U(x,t), && y=\infty . \end{aligned}$$

If with indices x, y, t, we denote the partial derivatives on respective coordinates and introduce the stream function $\psi(x,y,t)$ defined by

$$u=\psi_y, \qquad v=-\psi_x, \text{ with } \psi(x,0,t)\equiv 0,$$

the equation of the continuity (2.2) will be identically satisfied, and the equation (2.1) is reduced to the form

$$(2.4)\qquad \psi_{yt}+\psi_y\,\psi_{xv}-\psi_x\,\varphi_{yy}=U_t+UU_x+\nu\psi_{yyy},$$

with boundary conditions

$$(2.5)\qquad \begin{aligned} &\psi_y=U(x,t),\ \psi_x=0, && y=0 \text{ and } t=0,\\ &\psi_x=\psi_y=0, && y=0 \text{ and } t>0,\\ &\psi_x\to U(x,t), && y\to\infty . \end{aligned}$$

If the velocity of the external potential current is given in the form of

$$U(x,t)=V(x)\,\Omega(t),$$

and if $\Omega(t)$ is the function defined by $t\in[0,+\infty)$ and its square is integrable in some interval $[0,t]$ then, as we have already shown in the section 1, we can define the new variables, independent

$$(2.6)\qquad x, \quad \tau=\frac{1}{\nu}\int_0^t \Omega^2(t)\,dt, \quad \eta=\frac{\Omega(t)\,y}{\nu\sqrt{3\tau}},$$

and dependent

$$(2.7)\qquad \mathfrak{F}(x,\eta,\tau)=\frac{\psi(x,y,\tau)}{\nu\sqrt{3\tau}\,V(x)},$$

for every $x>0, y\geqslant 0$ and $t>0$.

Introducing expressions (2.6) and (2.7) in the equation (2.4) this will be transformed to the form

$$(2.8)\qquad \mathfrak{F}_{\eta\eta\eta}+\alpha(\tau)(1-\mathfrak{F}_\eta-\eta\,\mathfrak{F}_{\eta\eta})+\frac{3}{2}\eta\,\mathfrak{F}_{\eta\eta}-3\tau\,\mathfrak{F}_{\eta\tau}+\beta(\tau)\,[V'(1-\mathfrak{F}_\eta^2+\mathfrak{F}\mathfrak{F}_{\eta\eta})+ \\ +V(\mathfrak{F}_x\mathfrak{F}_{\eta\eta}-\mathfrak{F}_\eta\,\mathfrak{F}_{x\eta})]=0,$$

and the boundary conditions (2.5) to

$$(2.9)\qquad \begin{aligned} &\mathfrak{F}_\eta(x,0,\tau)=1, && t=0,\\ &\mathfrak{F}(x,0,\tau)=\mathfrak{F}_\eta(x,0,\tau)=0, && t>0,\\ &\lim_{\eta\to\infty}\mathfrak{F}_\eta(x,\eta,\tau)=1, \end{aligned}$$

where following notations are introduced

$$\alpha(\tau)=\frac{\dot{\Omega}\nu 3\tau}{\Omega^3}, \qquad \beta(\tau)=\frac{\nu 3\tau}{\Omega}. \tag{2.10}$$

Assuming that the function $V(x)$ is infinitely differentiable, then the solution of the partial equation (2.8) can be found in the form of a special series

$$\mathfrak{F}(x,\eta,\tau)=\mathfrak{F}_0(\eta,\tau)+V'(x)\,\mathfrak{F}_1(\eta,\tau)+V'^2(x)\,\mathfrak{F}_2(\eta,\tau)+ \\ +V(x)\,V''(x)\,\mathfrak{F}_{2a}(\eta,\tau)+\dots. \tag{2.11}$$

When in the solving of this problem the method of successive approximations is applied,which has its physical meaning in connection with the process of forming the boundary layer, then it is easily seen that such an iteration method yields to (2.11). For the time being it remains open the question of proving of convergence of this method, respectively of this series (2.11). On account of the function $V(x)$, which in itself contains the shape of the contour, namely, it represents the function which shows how the velocity $\Omega(t)$ changes on the plate, if it would be deformed in the contour of the arbitrary shape, we shall say something of the influence of the shape of the contour on the solidity of the convergence of the given method. Let us observe the two boundary cases of the contour: in the case of the plate which is put in the direction of the potential flow $V(x)=1$, and in the case of it when put vertically on the same flow $V(x)=x$. Then it is easy to see that in the first case $V'(x), V''(x), \dots = 0$, and in the second one $V'(x)=1$, $V''(x), \dots = 0$. It is obvious that in the first case the series (2.11) would reduce itself only to the term $\mathfrak{F}_0(\eta,\tau)$, that would represent the solution of the flowing on the plate with the potential flow of velocity $\Omega(t)$, while in the second case it would be reduced to

$$\mathfrak{F}(x,\eta,\tau)=\sum_{k=0}^{\infty}(V')^k\,\mathfrak{F}_k(\eta,\tau), \tag{2.12'}$$

or, as $V'(x)=1$ then to

$$\mathfrak{F}(x,\eta,\tau)=\sum_{k=0}^{\infty}\mathfrak{F}_k(\eta,\tau). \tag{2.12}$$

For the time being not taking account of coefficients $\mathfrak{F}_k(\eta,\tau)$, we can say that the convergence of this method is so much the better as the contour all the less deviates from the plate and so more the front edges of the contour are nearer to the null. In the case that the contour deviates more than is stated in the series (2.11), we should take a greater number of terms. Later, in the section 9 we shall return and explain this question a little fuller.

If the supposed solution (2.11) we put into the equation (2.18), then it will be separated in the system of partial equations

$$F_{\eta\eta\eta}+\alpha(\tau)(1-F_\eta-\eta F_{\eta\eta})+\frac{3}{2}\eta F_{\eta\eta}-3\tau F_{\eta\tau}=0,$$

$$\Phi_{\eta\eta\eta}-\alpha(\tau)(\Phi_\eta+\eta\Phi_{\eta\eta})+\frac{3}{2}\eta\,\Phi_{\eta\eta}-3\tau\,\Phi_{\eta\tau}+\beta(\tau)(1-F_\eta^2-FF_{\eta\eta})=0, \tag{2.13}$$

$$H_{\eta\eta\eta}-\alpha(\tau)(H_\eta+\eta H_{\eta\eta})+\frac{3}{2}\eta H_{\eta\eta}-3\tau H_{\eta\tau}+\beta(\tau)(-2F_\eta\Phi_\eta+ \\ +F\Phi_{\eta\eta}+\Phi F_{\eta\eta})=0,$$

$$H_{a\eta\eta\eta}-\alpha(\tau)(H_{a\eta}+\eta H_{a\eta\eta})+\frac{3}{2}\eta H_{a\eta\eta}-3\tau H_{a\eta\tau}+\beta(\tau)(\Phi F_{\eta\eta}-F_\eta\Phi_\eta)=0,$$

where are introduced the signs

$$\mathfrak{F}_0 = F,\ \mathfrak{F}_1 = \Phi,\ \mathfrak{F}_2 = H,\ \mathfrak{F}_{2a} = H_a, \ldots$$

The boundary conditions are

$$(2.14)\qquad \begin{aligned} &F_\eta = 1,\ \Phi_\eta = H_\eta = H_{a\eta} = \cdots = 0, && y=0 \text{ and } t=0,\\ &F = F_\eta = 0,\ \Phi = \Phi_\eta = 0, \ldots\ldots\ldots && y=0 \text{ and } t>0,\\ &F_\eta \to 1,\ \Phi_\eta \to 0, \ldots\ldots\ldots\ldots\ldots, && y\to\infty. \end{aligned}$$

In the above equations the functions $\alpha(\tau)$ and $\beta(\tau)$ have the central part, because the special data of every particular problem come explicitely to expression only through them.

We can express the function $\alpha(\tau)$ also in following way

$$(2.15)\qquad \alpha(\tau) = 3\tau \frac{d}{d\tau} \ln \frac{\Omega}{\Omega_0},$$

as $d\tau = \frac{\Omega^2}{\nu}$ dt. The constant velocity Ω_0 is introduced only on account of dimensional correctness.

From the third of expressions (2.6) one can see when $\Omega(t) \gtrless 0$ at $t>0$ then the variable τ grows monotonly. Therefore, the uniqueness of variables (2.6) is warranted everywhere, namely, τ is for instance uniquely expressed through t. Also $\alpha(\tau)$ is with τ at the given $\Omega(t)$ a unique function of t.

This function, on account of its central importance, we shall name in future *the local principal function.* At this, we have endeavoured that the term *principal* should be in concordance with the essentially same function in the paper [3].

The local principal function can be interpreted as *the local parameter of the form* [1], which can be easily shown. If we transform the thickness of displacement δ^* and the thickness of the drop of the impulse δ^{**} to the new variables, we obtain

$$(2.16)\qquad \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \frac{\nu\sqrt{3\tau}}{\Omega}\eta_0, \quad \text{where } \eta_0 = \lim_{\eta\to\infty} (\eta - \mathfrak{F}),$$

$$(2.17)\qquad \delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{\nu\sqrt{3\tau}}{\Omega} \int_0^\infty \mathfrak{F}_\eta (1 - \mathfrak{F}_\eta) d\eta.$$

Substituting (2.16) in the expression for the local principal function, we get

$$(2.18)\qquad \alpha(\tau) = \frac{1}{\eta_0^2} \frac{\delta^{*2}\dot{\Omega}}{\nu\Omega},$$

from there one can see that it really represents the local parameter of the form which appears it the paper [1].

The function $\beta(\tau)$ multiplied by V' would represent *the convective parameter of the form* [1]

$$(2.19)\qquad V'\beta(\tau) = \frac{\delta^{*2}\Omega}{\nu} V' = \frac{\delta^{*2} U_x}{\nu}.$$

The sum of these two parameters, local and convective ones, would give the entire parameter of the form [1], [6]

$$\alpha(\tau)+V'\beta(\tau)=\frac{\delta^{*2}(U_t+UU_x)}{\nu U}. \tag{2.20}$$

The process of development of the boundary layer, which has also formed the way for solution of the problem, has done that the central part has the local parameter of the form, i. e. its significance is primary, while the significance of the convective parameter of the form is secondary.

3. Connexions between the magnitudes of boundary layer and the newly introduced variables

For the thickness of displacement and the thickness of the drop of the impulse we have already found expressions in the function of new variables. The expression is also easily found for the skin friction

$$\frac{N(x,\tau)}{\rho\Omega_0^2}=U(x,t)\frac{\Omega(t)}{\Omega_0^2}\frac{1}{\sqrt{3\tau}}\mathfrak{F}_{\eta\eta}(x,0,\tau). \tag{3.1}$$

Now, we can find connexions $\Omega=\Omega(\tau)$ and $t=t(\tau)$. From the third of the function (2.10) we obtain

$$\Omega(t)=\Omega_0\exp\left(\frac{1}{3}\int_{\tau_0}^{\tau}\frac{\alpha(\tau)}{\tau}\,d\tau\right), \tag{3.2}$$

respectively, if we introduce the designation $f(\tau)=\exp(\quad)$ it follows

$$\Omega(\tau)=\Omega_0 f(\tau). \tag{3.3}$$

From the third of the equation (2.6) and the equation (3.2) we obtain

$$t-t_0=\frac{\nu}{\Omega_0^2}\int_{\tau_0}^{\tau}\frac{d\tau}{f^2(\tau)}. \tag{3.4}$$

From here, we can calculate in every special case the inverse function. For further work, which derives from the local principal function $\alpha(\tau)$ given in advance, we have obtained following expressions

$$u(x,y,t)=\Omega_0 V(x)f(\tau)\mathfrak{F}_{\eta}(x,\eta,\tau),$$

$$t-t_0=\frac{\nu}{\Omega_0^2}\int_{\tau_0}^{\tau}\frac{d\tau}{f^2(\tau)},\quad y=\frac{\nu}{\Omega_0}\frac{\sqrt{3\tau}}{f(\tau)}\eta. \tag{3.5}$$

In practical problems the function $\Omega(t)$, mostly will be given, and then it is necessary to determine the local principal function $\alpha(t)$. It will be given in the function of the time t, but from $\tau=\tau(t)$ by inversion $t=t(\tau)$, hence also $\alpha=\alpha(\tau)$.

In further work, we shall assume that the function $\Omega(t)$ is given in the form of unlimited series or in polynome on time t.

4. The development in series of function $\Omega(t), \alpha(\tau)$ and $\beta(t)$ and their connexions.

Let us suppose that the function $\Omega(t)$ is analytical on the interval $0 \leqslant t < \infty$ and it can be represented in term of convergent power series

(4.1) $$\Omega(t) = \sum_{k=0}^{\infty} \Omega_k t^k$$

We shall observe two cases of function $\Omega(t)$:

(4.2)
i) $$\Omega(t) = \Omega_0 + \Omega_1 t + \Omega_2 t^2 + \dots, \qquad \Omega_0 \neq 0,$$
ii) $$\Omega(t) = \Omega_1 t + \Omega_2 t^2 + \dots\dots, \qquad \Omega_1 \neq 0,$$

and give for them the form of local principal function.
The case I:

If a body is in movement at the velocity Ω_0, then the change of this state on time will be given by the law (4.2 i) respectively, it will belong to the case I.

Let us see now, what form will have the local principal function?

Substituting the expression (4.1 i) in the third of the equation (2.6) we shall obtain that

(4.3) $$\tau = \frac{1}{\nu}\left(\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{1}{k+i+1}\Omega_k\,\Omega_i\,t^{k+i+1}\right),$$

respectively

(4.3′) $$\tau = g_1 t + g_2 t^2 + \dots,$$

then from the theorem of inversion follows

(4.4) $$t = c_1\tau + c_2\tau^2 + \dots.$$

From (2.10) and (4.3) is

$$\alpha(\tau) = d_1 t + d_2 t^2 \dots,$$
$$\beta(\tau) = e_1 t + e_2 t^2 + \dots,$$

respectively

(4.5)
$$\alpha(\tau) = \alpha_1\tau + \alpha_2\tau^2 + \dots = \sum_{i=0}^{\infty}\alpha_i\tau^i, \text{ with } \alpha_0 = 0.$$
$$\beta(\tau) = \beta_1\tau + \beta_2\tau^2 + \dots = \sum_{i=1}^{\infty}\beta_i\tau^i.$$

If in (2.10) we introduce (4.1), (4.5) and the third of the expression (2.6) and if we equalize values along the degrees t, we shall obtain the coefficients α_i and β_i of functions $\alpha(\tau)$ and $\beta(\tau)$. Then follows:

$$\alpha_0 = 0$$

$$\alpha_1 = 3a_1$$

(4.6) $$\alpha_2 = -9a_1^2 + 6a_2$$

$$\alpha_3 = 27a_1^3 - 33a_1 a_2 + 9a_3$$

$$\alpha_4 = -81a_1^4 + 142a_1^2 a_2 - 54a_1 a_3 - 22a_2^2 + 12a_4$$

$$\alpha_5 = 243a_1^5 + \frac{2415}{10} a_1^2 a_3 + \frac{988}{5} a_1 a_2^2 - \frac{1093}{2} a_1^3 a_2 - 60a_2 a_3 - 81a_1 a_4 + 15a_5$$

(4.7)

$$\beta_0 = 0$$

$$\beta_1 = 3b_1$$

$$\beta_2 = -3b_2$$

$$\beta_3 = -3b_3 + 6b_1 a_1^2$$

$$\beta_4 = -3b_4 + 14b_2 a_2 - 14b_1 a_1^3$$

$$\beta_5 = -3b_5 + \frac{33}{2} b_4 a_1 - \frac{105}{2} b_3 a_1^2 + 7b_1 a_2^2 + 35b_1 a_1^4,$$

where

$$a_i = \frac{\Omega_i \nu^i}{\Omega_0^{2i+1}}, \qquad b_i = \frac{\Omega_{i-1} \nu^i}{\Omega_0^{2i}}.$$

If we introduce non-dimensional magnitudes

(4.8) $$\bar{t} = \frac{\Omega_0}{l} t \qquad \bar{\Omega}_k = \frac{\Omega_k}{\Omega_0^{k+1}} l^k,$$

and also we put

(4.9) $$\bar{\tau} = \frac{\tau}{R_e}, \quad \bar{\alpha}_k = \alpha_k R_e^k, \quad \bar{\beta}_k = \beta_k R_e^k,$$

then one sees that following relations exist

$$\Omega(t) = \sum_k \Omega_k t^k \leftrightarrow \bar{\Omega}(\bar{t}) = \sum_k \bar{\Omega}_k \bar{t}^k,$$

(4.10) $$\alpha(\tau) = \sum_k \alpha_k \tau_k \leftrightarrow \bar{\alpha}(\bar{\tau}) = \sum_k \bar{\alpha}_k \bar{\tau}^k,$$

$$\beta(\tau) = \sum_k \alpha_k \tau^k \leftrightarrow \bar{\beta}(\bar{\tau}) = \sum_k \bar{\beta}_k \bar{\tau}^k,$$

that means that systems of formulae remain formally the same. But, it is simpler to have magnitudes with signs.

The case II:

For this case, from (2.6) and (4.1) with respect to that $\Omega_0 = 0$ and $\Omega_1 \neq 0$, we get

$$\tau = \frac{1}{\nu} \left(\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k+i+1} \Omega_k \Omega_i t^{k+i+1} \right), \tag{4.11}$$

respectively

$$\tau = g_1 t^3 + g_2 t^4 + \cdots . \tag{4.11'}$$

By inversion we obtain

$$t = c_{1/3} \tau^{1/3} + c_{2/3} \tau^{2/3} + \cdots . \tag{4.12}$$

The function $\alpha(\tau)$ i $\beta(\tau)$ will have the following form

$$\alpha(\tau) = 1 + d_1 t + d_2 t^2 + \ldots ,$$

$$\beta(\tau) = e_2 t^2 + e_3 t^3 + \ldots ,$$

respectively, on account of (4.12)

$$\begin{aligned} \alpha(\tau) &= 1 + \alpha_{1/3} \tau^{1/3} + \cdots = \sum_{k=0}^{\infty} \alpha_{k/3} \tau^{k/3}, \text{ with } \alpha_0 = 1, \\ \beta(\tau) &= \beta_{2/3} \tau^{2/3} + \beta_1 \tau + \cdots = \sum_{k=2}^{\infty} \beta_{k/3} \tau^{k/3}. \end{aligned} \tag{4.13}$$

We can obtain easily the coefficients $\alpha_{k/3}$ and $\beta_{k/3}$ of these functions from (2.10), if we take into account the expressions (4.1), (4.13) and (2.6). It follows

$$\begin{aligned}
\alpha_0 &= 1 \\
\alpha_{1/3} &= \frac{1}{2} a_{1/3} \\
\alpha_{2/3} &= -\frac{23}{20} a_{1/3}^2 + \frac{6}{5} a_{2/3} \\
\alpha_1 &= \frac{203}{120} a_{1/3}^3 - \frac{296}{30} a_{1/3} a_{2/3} - \frac{4}{3} a_1 \\
\alpha_{4/3} &= -\frac{919}{600} a_{1/3}^4 + \frac{5226}{150} a_{1/3}^2 a_{2/3} + \frac{9}{7} a_{1/3} a_1 - \frac{1088}{175} a_{2/3}^2 - \frac{22}{21} a_{4/3} \\
\alpha_{5/3} &= -\frac{1839}{7200} a_{1/3}^5 - \frac{2931}{30} a_{1/3}^3 a_{2/3} + \frac{2004}{735} a_{1/3}^2 a_1 + \frac{9576}{175} a_{1/3} a_{2/3}^2 + \\
&\quad + \frac{539}{588} a_{1/3} a_{4/3} - \frac{39}{20} a_{2/3} a_1 - \frac{3}{4} a_{5/3},
\end{aligned} \tag{4.14}$$

$$
(4.15)\quad
\begin{aligned}
&\beta_0 = 0\\
&\beta_{1/3} = 0\\
&\beta_{2/3} = b_{2/3}\\
&\beta_1 = -\frac{1}{2} b_1\\
&\beta_{4/3} = \frac{1}{2} b_{2/3}\, a_{1/3}^2 - \frac{3}{5} b_{4/3}\\
&\beta_{5/3} = -\frac{7}{30} b_{2/3} a_{1/3}^3 + \frac{37}{30} b_1\, a_{2/3} - \frac{2}{3} b_{5/3},
\end{aligned}
$$

where

$$
a_{i/3} = \frac{\Omega_{i+1}\,(3\nu)^{i/3}}{\Omega_1^{2i/3+1}}, \qquad b_{i/3} = \frac{\Omega_{i-1}\,(3\nu)^{i/3}}{\Omega_1^{2i/3}}.
$$

Also here we can introduce non-dimensional magnitudes and show that the systems of formulae remain formally the same.

If the local principal function is given in advance then is easy to solve also the inverse problem i.e. to find the coefficients Ω_i of the function $\Omega(t)$ from the formulae for a_i and $a_{i/3}$.

5. Generalization of the given problem

Many of practical problems are comprised by the investigated cases I and II. But, it is possible to accomplish the generalization of the given problem, if we put the question, to which of external potential flows belong the local principal function given by the following forms

$$
(5.1)\quad
\begin{aligned}
&\text{i)}\quad \alpha(\tau) = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2 + \dots,\\
&\text{ii)}\quad \alpha(\tau) = \alpha_0 + \alpha_{1/3}\, \tau^{1/3} + \alpha_{2/3}\, \tau^{2/3} + \dots .
\end{aligned}
$$

In order to answer this question we shall return to the (3.2) in which instead of $\alpha(\tau)$ we shall substitute the above expressions i) and ii)

$$
(5.2)\quad
\begin{aligned}
&\text{i)}\quad \Omega(\tau) = \Omega_0 \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\alpha_0} (q_0 + q_1 \tau + q_2 \tau^2 + \cdots),\\
&\text{ii)}\quad \Omega(\tau) = \Omega_0 \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\alpha_0} (q_0 + q_{1/3}\, \tau^{1/3} + \cdots).
\end{aligned}
$$

In the same way for the given two values of the local principal function (5.1) from (3.4), we obtain

$$
(5.3)\quad
\begin{aligned}
&\text{i)}\quad t(\tau) = \tau^{1-\frac{2}{3}\alpha_0} (r_0 + r_1 \tau + \cdots),\\
&\text{ii)}\quad t(\tau) = \tau^{1-\frac{2}{3}\alpha_0} (r_0 + r_{1/3}\, \tau^{1/3} + \cdots).
\end{aligned}
$$

If we substitute (5.3) into (5.2) we obtain

$$\text{(5.4)}\qquad \begin{aligned} &\text{i) } \Omega(t) = t^m (s_0 + s_1 t^{2m+1} + s_2 t^{2(2m+1)} + \cdots), \\ &\text{ii) } \Omega(t) = t^m (s_0 + s_{1/3} t^{1/3(2m+1)} + \cdots), \end{aligned}$$

where

$$m = \frac{\alpha_0}{3 - 2\alpha_0}, \text{ respectively } \alpha_0 = \frac{3m}{2m+1}, \text{ at } m \neq -\frac{1}{2}.$$

From the general expression (5.4) it is easy to obtain the investigated cases I and II. So, for $\alpha_0 = 0\,(m=0)$ we obtain our case I, and for $\alpha_0 = 1$ $(m=1)$ our case II.

Let us observe still one special case of the local principal function

$$\text{(5.5)}\qquad \alpha(\tau) = \alpha_0$$

For this case from (3.2) we obtain

$$\text{(5.6)}\qquad \Omega(t) = \Omega_0 \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\alpha_0},$$

and from (3.4)

$$\text{(5.7)}\qquad t - t_0 = \frac{\nu}{\Omega_0^2} \int_{\tau_0}^{\tau} \left(\frac{\tau_0}{\tau}\right)^{\frac{2}{3}\alpha_0} d\tau = \begin{cases} \dfrac{3\nu\tau_0}{\Omega_0^2(3 - 2\alpha_0)} \left[\left(\dfrac{\tau_0}{\tau}\right)^{\frac{2}{3}\alpha_0 - 1} - 1\right] & \text{for } \alpha_0 \neq \dfrac{3}{2}, \\ \dfrac{\nu\tau_0}{\Omega_0^2} \ln\left(\dfrac{\tau}{\tau_0}\right) & \text{for } \alpha_0 = \dfrac{3}{2}. \end{cases}$$

By a suitable choice of the disponible value t_0, we can write this result as

$$\text{(5.8)}\qquad t = \begin{cases} t_0 \left(\dfrac{\tau_0}{\tau}\right)^{\frac{2}{3}\alpha_0 - 1} & \text{for } \alpha_0 \neq \dfrac{3}{2}, \\ t_0 \left[1 + \ln\left(\dfrac{\tau}{\tau_0}\right)\right] & \text{for } \alpha_0 = \dfrac{3}{2}. \end{cases}$$

From here it is easy to find the inverse function

$$\text{(5.9)}\qquad \frac{\tau}{\tau_0} = \begin{cases} \left(\dfrac{t}{t_0}\right)^{\frac{3}{3 - 2\alpha_0}} & \text{for } \alpha_0 \neq \dfrac{3}{2}, \\ exp\left(\dfrac{t - t_0}{t_0}\right) & \text{for } \alpha_0 = \dfrac{3}{2}. \end{cases}$$

If we substitute the given expression into (5.6), it follows

$$\Omega(t)=\begin{cases}\Omega_0\left(\dfrac{t}{t_0}\right)^m & \text{for } m\neq\infty\,,\\ \Omega_0\, exp\left(\dfrac{1}{3}\dfrac{t-t_0}{t_0}\right) & \text{for } m=\infty\,,\end{cases} \tag{5.10}$$

In the paper [2] it is shown that this form of velocity of the external potential flow on plate allows the similarity of solution and is simple to be solved, since the partial equation is reduced to the ordinary differential equation.

The function $\beta(\tau)$ will have the following form in cases (5.4i) and (5.4ii)

$$\begin{aligned} &\text{i)}\quad \beta(\tau)=\tau^k(\beta_0+\beta_1\tau+\beta_2\tau^2+\cdots),\\ &\text{ii)}\quad \beta(\tau)=\tau^k(\beta_0+\beta_{1/3}\tau^{1/3}+\beta_{2/3}\tau^{2/3}+\cdots),\end{aligned} \tag{5.11}$$

and in the case (5.10)

$$\beta(\tau)=\beta_0\tau^k, \tag{5.12}$$

where

$$k=\frac{m+1}{2m+1}.$$

6. The solution of the given system of partial equations

By choice of series for the modulated stream function $\mathfrak{F}(x,\eta,\tau)$ in the form (2.11) from the partial equation (2.8) we have obtained a system of the partial differential equations (2.13), then we have investigated in detail the connection between the function $\Omega(t)$ and the functions $\alpha(\tau)$ and $\beta(\tau)$ and in that way we have done all preparations for solution of the given system.

We shall investigate both of large classes i) and ii) of all local principal functions which in the observed interval $0\leqslant\tau<\tau_0$ can be developed in powers series

$$\begin{aligned} &\alpha(\tau)=\sum_{i=0}^{\infty}\alpha_i\tau^i,\\ &\alpha(\tau)=\sum_{i=0}^{\infty}\alpha_{1/3}\tau^{i/3},\end{aligned} \tag{6.1}$$

with an arbitrary (rational) α_0. Earlier we have seen that to them belong in uniform correspondence the external potential velocities of the forms

$$\begin{aligned} &\Omega(t)=\sum_{k=0}^{\infty}t^m\,\Omega_k\,t^{k(2m+1)},\\ &\Omega(t)=\sum_{k=0}^{\infty}t^m\,\Omega_{k/3}\,t^{k/3(2m+1)}.\end{aligned} \tag{6.2}$$

All values of the parameter $0 \leqslant m < \infty \left(0 \leqslant \alpha_0 < \frac{3}{2}\right)$ can be of interest in application. But, here we shall restrict ourselves only to two special cases to which the following values of the parameter: $m=0$ $(\alpha_0=0)$ and $m=1$ $(\alpha_0=1)$ correspond, and which are practically also most interesting. The case $\alpha_0=0$ we have denoted earlier as

The case I:

$$\alpha_0 = 0 \qquad (m=0),$$

$$\Omega(t) = \sum_{k=0}^{\infty} \Omega_k t^k,$$

with $\Omega_0 \neq 0$ (and otherwise arbitrarly), Ω_i – arbitrarly $(i=1,2\ldots)$.
The calculation of the coefficients α_k from the coefficients Ω_k is given with the equations (4.6), and the coefficients β_k with (4.7).

If in the system of partial equations (2.13) instead of $\alpha(\tau)$ and $\beta(\tau)$ we introduce the expressions (5.li) respectively (5.lii) then the given solutions will be suitable for calculation as powers series in τ with coefficients which are functions of the reduced distance η from the wall

$$F(\eta,\tau) = \sum_{k=0}^{\infty} F_k(\eta)\,\tau^k, \qquad \Phi(\eta,\tau) = \sum_{k=1}^{\infty} \Phi_k(\eta)\,\tau^k,$$

$$H(\eta,\tau) = \sum_{k=2}^{\infty} H_k(\eta)\,\tau^k.$$

Substituting such assumed solutions into the system (2.13) each of equations of this system will separate itself into the system of ordinary differential equations for determination of coefficients-functions of every of above powers series. As, the systems are recursive we shall write them in the form of recurrent formulea

$$F_0''' + \alpha_0\,(1 - F_0' - \eta\,F_0'') + \frac{3}{2}\eta\,F_0'' = 0,$$

$$F_k''' + \alpha_k\,(1 - F_0' - \eta\,F_0'') + \frac{3}{2}\eta\,F_k'' - 3k\,F_k' + \sum_{i=0}^{k-1} \alpha_i\,(F_{k-i}' + \eta\,F_{k-i}'') = 0,$$
$$(k=1,2,\ldots)$$

$$\Phi_1''' + \alpha_0\,(\Phi_1' + \eta\,\Phi_1'') + \frac{3}{2}\eta\,\Phi_1'' - 3\,\Phi_1' + \beta_1\,(1 - F_0'^2 + F_0\,F_0'') = 0,$$

(6.4)
$$\Phi_k''' - \sum_{i=0}^{k-1} \alpha_i\,(\Phi_{k-i}' + \eta\,\Phi_{k-i}'') + \frac{3}{2}\eta\,\Phi_k'' - 3k\,\Phi_k' + \beta_k\,(1 - F_0'^2 + F_0 F_0'') +$$
$$+ \sum_{i=1}^{k-1} \beta_i \left[- \sum_{j=0}^{k-i} F_j' F_{k-i-j}' + \sum_{j=0}^{k-i} F_j\,F_{k-i-j}'' \right] = 0, \qquad (k=2,3,\ldots)$$

$$H_k''' - \sum_{i=0}^{k-2} \alpha_i\,(H_{k-i}' + \eta H_{k-1}'') + \frac{3}{2}\eta\,H_k'' - 3k\,H_k' + \sum_{i=1}^{k-i} \beta_i \left[-2 \sum_{j=1}^{k-1} F_{k-i-j}'\,\Phi_j' + \right.$$
$$\left. + \sum_{j=1}^{k-i} F_{k-i-j}\,\Phi_j'' + \sum_{j=i}^{k-i} F_{k-i-j}''\,\Phi_j \right] = 0, \qquad (k=2,3,\ldots)$$

$$H_{ak}''' - \sum_{i=0}^{k-2} \alpha_i (H_{ak-i}' + \eta H_{ak-i}'') + \frac{3}{2} \eta H_{ak}'' - 3 H_{ak}' + \sum_{i=1}^{k-1} \beta_i \left[\sum_{j=1}^{k-i} F_j'' \Phi_{k-i-j} - \right.$$
$$\left. - \sum_{j=1}^{k-i} F_j' \Phi_{k-i-j}' \right] = 0, \quad (k = 2,3, \ldots)$$

The boundary conditions transform themselves into the following forms

$$\begin{aligned}
&F_0'(0) = 0, \quad F_k(0) = F_k'(0) = 0 \\
&F_0'(\infty) = 1, \quad F_k'(\infty) = 0, \quad (k = 1,2, \ldots), \\
&\Phi_k(0) = \Phi_k'(0) = 0, \\
(6.5) \quad &\\
&\Phi_k'(\infty) = 0, \quad (k = 1,2, \ldots), \\
&H_k(0) = H_k'(0) = 0, \\
&H_k'(\infty) = 0, \quad (k = 2,3, \ldots).
\end{aligned}$$

In all given differential equations (6.4) appear the parameters $\alpha_0, \alpha_1, \ldots$ and $\beta_1, \beta_2, \ldots$ for determination of $F_k(\eta)$, $\Phi_k(\eta), \ldots$ For the case of similar solutions, we have had it, that the local principal function $\alpha(\tau)$ is reduced to α_0, and of the system of differential equations remains the equation for determination of $F_0(\eta)$. In this differential equation appears also α_0 except the variable η so then $F_0 = F_0(\eta, \alpha_0)$. As $F_0(\eta, \alpha_0)$ is meritory for calculation of all $F_k(\eta) \ldots$ $(k = 1,2, \ldots)$, then we cannot have functions which are independent of α_0 by giving a suitable form for $F_0(\eta), \ldots$ This one cannot expect either from physical point of view, because α_0 and $F_0(\eta, \alpha_0)$ that regulate the beginning profile which are decisive for further behaviour at streaming.

But, it is possible to express $F_k(\eta), \ldots$ through linear combination functions which are independent of $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ Then for any fixed value of α_0, these universal functions can be tabulated once for all.

Reducing $F_k(\eta), \ldots$ for any fixed value α_0 to universal functions we achieve it in the following way

$$\begin{aligned}
&F_0 = F_0 \\
&F_1 = \alpha_1 f_1 \\
&F_2 = \alpha_1^2 f_{11} + \alpha_2 f_2 \\
&F_3 = \alpha_1^3 f_{111} + \alpha_1 \alpha_2 f_{12} + \alpha_3 f_3 \\
&F_4 = \alpha_1^4 f_{1111} + \alpha_1^2 \alpha_2 f_{112} + \alpha_1 \alpha_3 f_{13} + \alpha_2^2 f_{22} + \alpha_4 f_4 \\
&F_5 = \alpha_1^5 f_{11111} + \alpha_1^3 \alpha_2 f_{1112} + \alpha_1^2 \alpha_3 f_{113} + \alpha_1 \alpha_2^2 f_{122} + \alpha_1 \alpha_4 f_{14} + \alpha_2 \alpha_3 f_{23} + \alpha_5 f_5 \\
(6.6) \quad &\Phi_1 = \beta_1 \varphi_1 \\
&\Phi_2 = \alpha_1 \beta_1 \varphi_{11} + \beta_2 \varphi_2 \\
&\Phi_3 = \alpha_1^2 \beta_1 \varphi_{111} + \alpha_1 \beta_2 \varphi_{12} + \alpha_2 \beta_1 \varphi_{12} + \beta_3 \varphi_3
\end{aligned}$$

$$\Phi_4 = \alpha_1^3\,\beta_1\,\varphi_{1111} + \alpha_1\,\alpha_2\,\beta_1\,\varphi_{121} + \alpha_1^2\,\beta_2\,\varphi_{112} + \alpha_1\,\beta_3\,\varphi_{13} + \alpha_3\,\beta_1\,\varphi_{31} + \\ + \alpha_2\,\beta_2\,\varphi_{22} + \beta_4\,\varphi_4$$

$$\Phi_5 = \alpha_1^4\,\beta_1\,\varphi_{11111} + \alpha_1^2\,\alpha_2\,\beta_1\,\varphi_{1121} + \alpha_1^3\,\beta_2\,\varphi_{1112} + \alpha_1\,\alpha_2\,\beta_2\,\varphi_{122} + \alpha_1^2\,\beta_3\,\varphi_{113} + \\ + \alpha_1\,\alpha_3\,\beta_1\,\varphi_{131} + \alpha_1\,\beta_4\,\varphi_{14} + \alpha_2\,\beta_3\,\varphi_{23} + \alpha_2^2\,\beta_1\,\varphi_{221} + \alpha_4\,\beta_{31}\,\varphi_{41} + \alpha_3\,\beta_2\,\varphi_{32} + \beta_5\,\alpha_5$$

$$H_2 = \beta_1^2\,h_{11}$$

$$H_3 = \alpha_1\,\beta_1^2\,h_{111} + \beta_1\,\beta_2\,h_{12}$$

$$H_4 = \alpha_1^2\,\beta_1^2\,h_{1111} + \alpha_2\,\beta_1^2\,h_{211} + \alpha_1\,\beta_1\,\beta_2\,h_{112} + \beta_1\,\beta_3\,h_{13} + \beta_2^2\,h_{22}$$

$$H_5 = \alpha_1^3\,\beta_1^2\,h_{11111} + \alpha_1\,\alpha_2\,\beta_1^2\,h_{1211} + \alpha_3\,\beta_1^2\,h_{311} + \alpha_1^2\,\beta_1\,\beta_2\,h_{1112} + \alpha_1\,\beta_2^2\,h_{122} + \\ + \alpha_2\,\beta_1\,\beta_2\,h_{212} + \alpha_1\beta_1\,\beta_3\,h_{113} + \beta_2\,\beta_3\,h_{23} + \beta_1\,\beta_4\,h_{14}$$

Therefore, as stated above, the new systems of equations for determination of unknown $f_1, \ldots \varphi_1, \ldots$ will be independent of coefficients $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ All differential equations and boundary conditions are in Section 9.
The case II:

$$\alpha_0 = 1 \quad (m = 1),$$

$$\Omega(\tau) = \sum_{k=0}^{\infty} \Omega_{k+1}\, t^{k+1},$$

with $\Omega_1 \neq 0$ (and otherwise arbitrarly), Ω_i – arbitrarly ($i = 2, 3, \ldots$)· The calculation of coefficients $\alpha_{k/3}$ from Ω_k is given with equations (4.14) and the coefficients $\beta_{k/3}$ with (4.15).

Taking into account the forms of functions $\alpha(\tau)$ and $\beta(\tau)$ which are given by equations (5.11i) respectively (5.11 ii) the given solutions will be suitable for calculation as powers series in $\tau^{1/3}$ with coefficients depending on η

$$F(\eta, \tau) = \sum_{k=0}^{\infty} F_{k/3}(\eta)\,\tau^{k/3}, \qquad \Phi(\eta, \tau) = \sum_{k=2}^{\infty} \Phi_{k/3}(\eta)\,\tau^{k/3},$$

(6.8)

$$H(\eta, \tau) = \sum_{k=4}^{\infty} H_{k/3}(\eta)\,\tau^{k/3}.$$

The systems (2.13) because of (6.8) will be separated to following recursive systems of the ordinary differential equations

$$F_0''' + \alpha_0\left(1 - F_0' - \eta F_0''\right) + \frac{3}{2}\eta\, F_0'' = 0,$$

$$F_{k/3}''' + \alpha_{k/3}\left(1 - F_0' - \eta\, F_0''\right) + \frac{3}{2}\eta\, F_{k/3}'' - k\, F_{k/3}' -$$

$$- \sum_{i=0}^{k-1} \alpha_{i/3}\left(F_{\frac{k-i}{3}}' + \eta\, F_{\frac{k-i}{3}}''\right) = 0, \qquad (k = 1, \ldots)$$

(6.9)

$$\Phi_{2/3}''' - \alpha_0\left(\Phi_{2/3}' + \eta\, \Phi_{2/3}''\right) + \frac{3}{2}\eta\, \Phi_{2/3}'' - 2\,\Phi_{2/3}' + \beta_{2/3}\left(1 - F_0'^2 + F_0 F_0''\right) = 0,$$

$$\Phi'''_{k/3}-\sum_{i=0}^{k-2}\alpha_{i/3}\left(\Phi'_{\frac{k-i}{3}}+\eta\,\Phi''_{\frac{k-i}{3}}\right)+\frac{3}{2}\eta\,\Phi''_{k/3}-k\,\Phi'_{k/3}+$$
$$+\beta_{k/3}\left(1-F_0'^2+F_0F_0''\right)+\sum_{i=2}^{k-1}\beta_{i/3}\left[-\sum_{j=0}^{k-i}F'_{j/3}F'_{\frac{k-i-j}{3}}+\right.$$
$$\left.+\sum_{j=0}^{k-i}F_{j/3}F''_{\frac{k-i-j}{3}}\right]=0,\qquad (k=3,\ldots)$$

$$H'''_{k/3}-\sum_{i=0}^{k-4}\alpha_{i/3}\left(H'_{\frac{k-i}{3}}+\eta\,H''_{\frac{k-i}{3}}\right)+\frac{3}{2}\eta\,H''_{k/3}-k\,H'_{k/3}+\sum_{i=2}^{k-2}\beta_{i/3}\Bigg[$$
$$\left[-2\sum_{j=0}^{k-i-2}F'_{\frac{k-i-j}{3}}\Phi'_{j/3}+\sum_{j=0}^{k-i-2}F_{\frac{k-i-j}{3}}\Phi''_{j/3}+\sum_{j=0}^{k-i-2}F''_{\frac{k-i-j}{3}}\Phi_{j/3}\right]=0\quad (k=4,\ldots$$

$$H'''_{a\,k/3}-\sum_{i=0}^{k-4}\alpha_{i/3}\left(H'_{a\frac{k-i}{3}}+\eta\,H''_{a\frac{k-1}{3}}\right)+\frac{3}{2}\eta\,H''_{k/3}-k\,H'_{k/3}+\sum_{i=2}^{k-2}\beta_{i/3}\Bigg[$$
$$\left[\sum_{j=0}^{k-i-2}F''_{j/3}\Phi_{\frac{k-i-j}{3}}-\sum_{j=0}^{k-i-2}F'_{j/3}\Phi'_{\frac{k-i-j}{3}}\right]=0,\qquad (k=4,\ldots)$$

The boundary conditions are

$$F_0'(0)=0,\qquad F_{k/3}(0)=F'_{k/3}(0)=0,$$
$$F_0'(\infty)=1,\qquad F'_{k/3}(\infty)=0,\qquad (k=1,2,\ldots)$$

(6.10)

$$\Phi_{k/3}(0)=\Phi'_{k/3}(0)=0,$$
$$\Phi'_{k/3}(\infty)=0,\qquad (k=2,3,\ldots)$$

. . . .

The reduction of $F_{k/3}(\eta),\ldots$ for any fixed α_0 on universal functions will be achieved in the following way

$$F_0=F_0$$
$$F_{1/3}=\alpha_{1/3}f_{1/3}$$
$$F_{2/3}=\alpha_{1/3}^2f_{1/3\,1/3}+\alpha_{2/3}f_{2/3}$$
$$F_1=\alpha_{1/3}^3f_{1/3\,1/3\,1/3}+\alpha_{1/3}\alpha_{2/3}f_{1/3\,2/3}+\alpha_1f_1$$
$$F_{4/3}=\alpha_{1/3}^4f_{1/3\,1/3\,1/3\,1/3}+\alpha_{1/3}^2\alpha_{2/3}f_{1/3\,1/3\,2/3}+\alpha_{1/3}\alpha_1f_{1/3\,\,1}+$$
$$+\alpha_{2/3}^2f_{2/3\,2/3}+\alpha_{4/3}f_{4/3}$$
$$F_{5/3}=\alpha_{1/3}^5f_{1/3\,1/3\,1/3\,1/3\,1/3}+\alpha_{1/3}^3\alpha_{2/3}f_{1/3\,1/3\,1/3\,2/3}+\alpha_{1/3}^2\alpha_1f_{1/3\,1/3\,1}+$$
$$+\alpha_{1/3}\alpha_{2/3}^2f_{1/3\,2/3\,2/3}+\alpha_{1/3}\alpha_{4/3}f_{1/3\,4/3}+\alpha_{2/3}\alpha_1f_{2/3\,1}+\alpha_{5/2}f_{5/3},$$

(6.11)

$$\Phi_{2/3}=\beta_{2/3}\varphi_{2/3}$$
$$\Phi_1=\alpha_{1/3}\beta_{2/3}\varphi_{1/3\,2/3}+\beta_1\varphi_1$$
$$\Phi_{4/3}=\alpha_{1/3}^2\beta_{2/3}\varphi_{1/3\,1/3\,2/3}+\alpha_{2/3}\beta_{2/3}\varphi_{2/3\,2/3}+\alpha_{1/3}\beta_1\varphi_{1/3\,\,1}+\beta_{4/3}\varphi_{4/3}$$

$$\Phi_{5/3}=\alpha_{1/3}^{3}\beta_{2/3}\varphi_{1/3\,1/3\,1/3\,2/3}+\alpha_{1/3}\alpha_{2/3}\beta_{2/3}\varphi_{1/3\,2/3\,2/3}+\alpha_1\beta_{2/3}\varphi_{1\,2/3}+$$
$$+\alpha_{1/3}^{2}\beta_1\varphi_{1/3\,1/3\,1}+\alpha_{2/3}\beta_1\varphi_{2/3\,1}+\alpha_{1/3}\beta_{4/3}\varphi_{1/3\,4/3}+\beta_{5/3}\varphi_{5/3},$$

$$H_{4/3}=\beta_{2/3}^{2}h_{2/3\,2/3}$$

$$H_{5/3}=\alpha_{1/3}\beta_{2/3}^{2}h_{1/3\,2/3\,2/3}+\beta_{2/3}\beta_1 h_{2/3\,1}.$$

All differential equations and boundary conditions are in Section 9
The special case:

$$\Omega(t)=\Omega_0 t^m.$$

The functions $\alpha(\tau)$ and $\beta(\tau)$ have the forms

$$\alpha(\tau)=\alpha_0,$$

(6.13)

$$\beta(\tau)=\beta\tau^k,$$

where

$$\beta=\frac{3}{2m+1}\Omega_0\left[\frac{\nu(2m+1)}{\Omega_0^2}\right]^{\frac{m+1}{2m+1}}=\frac{3}{2m+1}\beta^*.\quad k=\frac{m+1}{2m+1}.$$

From the expression (6.13) it is to be seen that the solutions $F(\eta,\tau),\ldots$ ought to be found in the following form

(6.14) $$F(\eta,\tau)=F(\eta),\quad \Phi(\eta,\tau)=\Phi(\eta)\tau^k,\quad H(\eta,\tau)=H(\eta)\tau^{2k}.$$

In this case the indepedent variables η and τ are reduced to the form

(6.15) $$\eta=2\sqrt{\frac{2m+1}{3}}\eta_1,\qquad \tau=\frac{1}{\beta^*}t^{2m+1},$$

where:

η_1 — the variable of similar solutions.

If we introduce the combinations

$$F=F_0,\quad \Phi=\beta^*\varphi,\quad H=\beta^{*2}h,$$

then after a very few elementary calculations the system of partial differential equations (2.13) will be reduced to the following system of ordinary differential equations

$$F_0'''+2\eta_1F_0''-4m(1-F_0')=0,$$

$$\varphi'''+2\eta_1\varphi''-4(2m+1)\varphi'=-4(1-F_0'^2+F_0F_0''),$$

(6.16)

$$h'''+2\eta_1h''-4(3m+2)h'=-4(-2F_0'\varphi'+F_0\varphi''+F_0''\varphi),$$

$$h_a'''+2\eta_1h_a''-4(3m+2)h_a'=-4(F_0''\varphi-F_0'\varphi'),$$

. . . .

with boundary conditions

$$
(6.17)\qquad
\begin{array}{lll}
F_0(0)=F_0'(0)=0, & \varphi(0)=\varphi'(0)=0 & h(0)=h'(0)=0,\dots \\
F_0'(\infty)=1, & \varphi'(\infty)=0, & h'(\infty)=0,\dots
\end{array}
$$

From (2.7) taking into account (2.11) and (6.14) for the stream function $\psi(x, y, t)$ we shall obtain the following expression

$$
\psi\,(x, y, t)=2\sqrt{\nu t}\,V(x)\,\Omega_0 t^m \{F_0(\eta_1)+\Omega_0\,V' t^{m+1}\varphi(\eta_1)+\Omega_0^2 t^{2(m+1)}[\,[V^{12} h(\eta_1)+VV''\,h_a(\eta_1)\,]+\cdots\}.
$$

Obviously, the equations (6.16) as well as the shape of stream function (6.18) are quite the same with Watson's [10], which he has obtained by solving the same problem.

7. Finding the time separation and the way which the body has performed in a given time

With unsteady boundary layers it is interesting to find the moment of beginning of separation from the contour, respectively the time interval which it takes to the moment when it begins the first time to be separated. From the first moment of the appearance of separation, the point of separation will move along the contour to the attainment of the steady state.

The position of separation at different moment of time can be calculated from the condition

$$N_{tan}=0$$

from where

$$\mathfrak{F}_{\eta\eta}\,(x, o, t)=0,$$

respectively

$$
(7.1)\qquad \sum_{k=0}^{\infty} F_k''(0)\,\tau^k+V'\sum_{k=1}^{\infty}\Phi_k''(0)\,\tau^k+V'^2\sum_{k=2}^{\infty}H_k''(0)\,\tau^k+ + VV''\sum_{k=2}^{\infty}H_{ak}''(0)\,\tau^k+\cdots=0
$$

From here in each special case, stopping at the determined term of the special series and with determined terms τ of power series in the region of convergence, we can find the position of the separation point on the contour at different moment of time.

The way which the body traverses at different time intervals we can find from the expression

$$
(7.2)\qquad s=\int_0^t \Omega(\tau)\,dt.
$$

8. The way for application of the method

We shall give instructions here for application of the given method in concrete cases. Normally, the problems of theory of boundary layers are so put up that outer velocity distribution is a prescribed one, moreover in the same way the local principal function $\alpha(\tau)$ can be given in advance. We shall take the first more normally case and on Case I show the application of it. In Case I:

$$\Omega(t)=\Omega_0+\Omega_1 t+\Omega_2 t^2+\cdots,$$

respectively

(8.1) $$U(x,t)=V(x)(\Omega_0+\Omega_1 t+\cdots).$$

To attain a more easy work it is better all magnitudes to make non-dimensional

(8.2)
$$\bar{x}=\frac{x}{l},\qquad \bar{t}=\frac{\Omega_0}{l}t,\qquad \bar{y}=y\frac{\sqrt{R_e}}{l},\qquad R_e=\frac{\Omega_0 l}{\nu},$$
$$\bar{U}(\bar{x},\bar{t})=\frac{U(x,t)}{\Omega_0},\qquad \bar{V}=V,\ \bar{V}'(\bar{x})=V'l,\qquad \bar{\Omega}(\bar{t})=\frac{\Omega(t)}{\Omega_0},$$
$$\bar{\alpha}_k=\alpha_k R_e^k,\quad \bar{\beta}_k=\beta_k R_e^k,\ \bar{F}_k=F_k R_e^k,\ \bar{\Phi}_k=\Phi_k\frac{1}{l}R_e^k,\quad \bar{H}_k=H_k\frac{1}{l^2}R_e^k\cdots$$

1. First of all one has to calculate the transition from coordinates y, t to the coordinates

(8.3) $$\bar{\tau}=\int_0^{\bar{t}}\bar{\Omega}^2(\bar{t})\,d\bar{t},\quad \eta=\frac{\bar{\Omega}(\bar{t})}{\sqrt{3\bar{\tau}}}\bar{y}=g(\bar{t})\bar{y},$$

and since the function $g(\bar{t})$ appears also later it is useful to evaluate it separately.

2. From the given coefficients $\bar{\Omega}_k$ it is easy to calculate the coefficients $\bar{\alpha}_k$ and $\bar{\beta}_k$ through the formulae (4.6) and (4.7).

3. The wall shearing stress is calculated from the formule

(8.4) $$\frac{N(\bar{x},\bar{t})}{\rho\Omega_0^2}R_e^{1/2}=\bar{U}(\bar{x},\bar{t})\,g(\bar{t})\Big[\sum_{k=0}\bar{F}_k''(0)\bar{\tau}^k+\bar{V}'(\bar{x})\sum_{k=1}\bar{\Phi}_k''(0)\bar{\tau}^k+\ldots\Big]$$

4. The displacement thickness of the boundary layer is easily obtained from (2.16)

(8.5) $$\frac{\delta^*(\bar{x},\bar{t})R_e^{1/2}}{l}=\frac{1}{g(\bar{t})}\Big\{\lim_{\eta\to\infty}(\eta-\bar{F}_0)-\Big[\sum_{k=0}\bar{F}_k''(\infty)\bar{\tau}^k+ +\bar{V}'(\bar{x})\sum_{k=1}\bar{\Phi}_k''(\infty)\bar{\tau}^k+\cdots\Big]\Big\}$$

5. The velocity profiles on different places along the contour and in different moments of time is calculated from the formulae

(8.6) $$\frac{u(\bar{x},\bar{y},\bar{t})}{\Omega_0}=\bar{U}(\bar{x},\bar{t})\Big[\sum_{k=0}\bar{F}_k'(\eta)\bar{\tau}^k+\bar{V}'(\bar{x})\sum_{k=1}\bar{\Phi}_k'(\eta)\bar{\tau}^k+\cdots\Big].$$

6. To find the separation moment we use the expression

(8.7) $$\sum_{k=0}\bar{F}_k''(0)\bar{\tau}^k+\bar{V}'(\bar{x})\sum_{k=1}\bar{\Phi}_k''(0)\bar{\tau}^k+\cdots=0.$$

7. The expression for the way which the body traverses in a certain time we obtain from

$$(8.8) \qquad \bar{s}=\int_0^{\bar{t}} \overline{\Omega}(\bar{t})\, d\bar{t}.$$

8. As the attainment of the first contourn connexion is necessary to the exact solution, then it is useful to verify it here too.
From the condition

$$(8.9) \qquad U_t + U U_x = -\nu \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0},$$

with respect to

$$u = U(x, t) \left[\sum_{k=0} F_k'(\eta)\, \tau^k + V'(x) \sum_{k=1} \Phi_k'(\eta)\, \tau^k + \cdots \right],$$

we obtain

$$(8.10) \qquad \begin{aligned} \dot{\Omega} &= -\nu\, \Omega g^2 \sum_{k=0} F_k'''(0)\, \tau^k, \\ \Omega &= -\nu g^2 \sum_{k=1} \Phi_k'''(0)\, \tau^k, \\ 0 &= \sum_{k=2} \overline{H}_k'''(0)\, \tau^k, \\ 0 &= \sum_{k=2} H_{ak}'''(0)\, \tau^k, \end{aligned}$$

The above expressions could be transformed further if one takes into account that

$$\bar{\alpha}(\bar{\tau}) = \frac{\dot{\overline{\Omega}}}{\overline{\Omega}\, \bar{g}^2}, \qquad \bar{\beta}(\bar{\tau}) = l\, \frac{\overline{\Omega}}{\bar{g}^2},$$

to the form

$$(8.11) \qquad \begin{aligned} &\sum_{k=0} \overline{F}_k'''(0)\, \bar{\tau}^k = -\bar{\alpha}(\bar{\tau}), \\ &\sum_{k=1} \overline{\Phi}{}^*{}_k'''(0)\, \bar{\tau}^k = -\bar{\beta}(\bar{\tau}), \\ &\sum_{k=1} \overline{H}_k'''(0)\, \bar{\tau}^k = 0, \\ &\sum_{k=2} \overline{H}_{ak}'''(0)\, \bar{\tau}^k = 0 \end{aligned}$$

where

. . . .

$$\overline{\Phi}{}^* = l\overline{\Phi}.$$

9. The solution of systems of equations for determination of universal functions

If we define the linear operator with expression

$$L_k(F) = F''' - \alpha_0 (F' + \eta F'') + \frac{3}{2} \eta F'' - 3kF', \tag{9.1}$$

then the systems of differential equations for determination of universal functions $f \ldots (\eta)$, $\varphi \ldots (\eta), \ldots$ could be written in the form

The case I:

$$\begin{aligned}
k=0 &\quad L_0(F_0) = 0 \\
k=1 &\quad L_1(f_1) = -(1 - F_0' - \eta F_0'') \\
k=2 &\quad L_2(f_{11}) = f_1' + \eta f_{11}'' \\
&\quad L_2(f_2) = -(1 - F_0' - \eta F_0'') \\
k=3 &\quad L_3(f_{111}) = f_{11}' + \eta f_{11}'' \\
&\quad L_3(f_{12}) = f_2' + \eta f_2' + f_{11}' + \eta f_{11}'' \\
&\quad L_3(f_3) = -(1 - F_0' - \eta F_0'') \\
k=4 &\quad L_4(f_{1111}) = f_{111}' + \eta f_{111}'' \\
&\quad L_4(f_{112}) = f_{12}' + \eta f_{12}'' + f_{11}' + \eta f_{11}'' \\
&\quad L_4(f_{13}) = f_3' + \eta f_3'' + f_1' + \eta f_1'' \\
&\quad L_4(f_{22}) = f_2' + \eta f_2'' \\
&\quad L_4(f_4) = -(1 - F_0' - \eta F_0'') \\
k=5 &\quad L_5(f_{11111}) = f_{1111}' + \eta f_{1111}'' \\
&\quad L_5(f_{1112}) = f_{112}' + \eta f_{112}'' + f_{111}' + \eta f_{111}'' \\
&\quad L_5(f_{113}) = f_{13}' + \eta f_{13}'' + f_{11}' + \eta f_{11}'' \\
&\quad L_5(f_{122}) = f_{22}' + \eta f_{22}'' + f_{12}' + \eta f_{12}'' \\
&\quad L_5(f_{14}) = f_4' + \eta f_4'' + f_1' + \eta f_1'' \\
&\quad L_5(f_{23}) = f_3' + \eta f_3'' + f_2' + \eta f_2'' \\
&\quad L_5(f_5) = -(1 - F_0' - \eta F_0'')
\end{aligned}$$

(9.2)

$$\begin{aligned}
k=1 &\quad L_1(\varphi_1) = -(1 - F_2'^2 + F_0 F_0'') \\
k=2 &\quad L_2(\varphi_{11}) = \varphi_1' + \eta \varphi_1'' - (-2F_0' f_1' + F_0 f_1'' + f_1 F_0'') \\
&\quad L_2(\varphi_2) = -(1 - F_0'^2 + F_0 F_0'') \\
k=3 &\quad L_3(\varphi_{111}) = \varphi_{11}' + \eta \varphi_{11}'' - (-2F_0' f_{11}' - f_1'^2 + F_0 f_{11}'' + f_1 f_1'' + f_{11} F_0'')
\end{aligned}$$

$$L_3(\varphi_{12})=\varphi_2'+\eta\varphi_2''-\left(-2F_0'f_1'+F_0f_1''+f_1F_0''\right)$$

$$L_3(\varphi_{21})=\varphi_1'+\eta\varphi_1''-\left(-2F_0'f_2'+F_0f_2''+f_2F_0''\right)$$

$$L_3(\varphi_3)=-\left(1-F_0'^2+F_0F_0''\right)$$

$k=4$

$$L_4(\varphi_{1111})=\varphi_{111}'+\eta\varphi_{111}''-\left(-2F_0'f_{111}'-2f_1'f_{11}'+F_0f_{111}''+\right.$$
$$\left.+f_1f_{11}''+f_{11}f_1''+f_{111}F_0''\right)$$

$$L_4(\varphi_{121})=\varphi_{21}'+\eta\varphi_{21}''+\varphi_{11}'+\eta\varphi_{11}''-\left(-2F_0'f_{12}'-2f_1'f_2'+\right.$$
$$\left.+F_0f_{12}'+f_1f_2''+f_2f_1''+f_{12}F_0''\right)$$

$$L_4(\varphi_{112})=\varphi_{12}'+\eta\varphi_{12}''-\left(-2F_0'f_{11}'-f_1'^2+F_0f_{11}''+f_1f_1''+f_{11}F_0''\right)$$

$$L_4(\varphi_{31})=\varphi_1'+\eta\varphi_1''-\left(-2F_0'f_3'+F_0f_3''+f_3F_0''\right)$$

$$L_4(\varphi_{22})=\varphi_2'+\eta\varphi_2''-\left(-2F_0'f_2+F_0f_2''+f_2F_0''\right)$$

$$L_4(\varphi_4)=-\left(1-F_0'^2+F_0F_0''\right)$$

$k=5$

$$L_5(\varphi_{11111})=\varphi_{1111}'+\eta\varphi_{1111}''-\left(-2F_0'f_{1111}'-2f_1'f_{111}'-f_{11}'^2+F_0f_{1111}''+\right.$$
$$\left.+f_1f_{111}''+f_{11}f_{11}''+f_{111}f_1''+f_{1111}F_0''\right)$$

$$L_5(\varphi_{1121})=\varphi_{121}'+\eta\varphi_{121}''+\varphi_{111}'+\eta\varphi_{111}''-\left(-2F_0'f_{112}-\right.$$
$$-2f_1'f_{12}'-2f_{11}'f_2'+F_0f_{112}''+f_1f_{12}''+f_2f_{11}''+f_{11}f_2''+$$
$$\left.+f_{12}f_1''+f_{112}F_0''\right)$$

$$L_5(\varphi_{131})=\varphi_{31}'+\eta\varphi_{31}''+\varphi_{11}'+\eta\varphi_{11}''-\left(-2F_0'f_{13}-2f_1'f_3'+\right.$$
$$\left.+F_0f_{13}''+f_1f_3''+f_3f_1''+f_{13}F_0''\right)$$

$$L_5(\varphi_{221})=\varphi_{21}'+\eta\varphi_{21}''-\left(-2F_0'f_{22}'-f_2'^2+F_0f_{22}''+f_2f_2''+f_{22}F_0''\right)$$

$$L_5(\varphi_{1112})=\varphi_{112}'+\eta\varphi_{112}''-\left(-2F_0'f_{111}'-2f_1'f_{11}'+F_0f_{111}''+\right.$$
$$\left.+f_1f_{11}''+f_{11}f_1''+f_{111}F_0''\right)$$

$$L_5(\varphi_{122})=\varphi_{22}'+\eta\varphi_{22}''+\varphi_{12}'+\eta\varphi_{12}''-\left(-2F_0'f_{12}-2f_1'f_2'+\right.$$
$$\left.+F_0f_{12}''+f_1f_2''+f_2f_1''+f_{12}F_0''\right)$$

$$L_5(\varphi_{113})=\varphi_{13}'+\eta\varphi_{13}''-\left(-2F_0'f_{11}'-f_1'^2+F_0f_{11}''+f_1f_1''+f_{11}F_0''\right)$$

$$L_5(\varphi_{32})=\varphi_2'+\eta\varphi_2''-\left(-2F_0'f_3'+F_0f_3''+f_3F_0''\right)$$

$$L_5(\varphi_{23})=\varphi_3'+\eta\varphi_3''-\left(-2F_0'f_2'+F_0f_2''+f_2F_0''\right)$$

$$L_5(\varphi_{41})=\varphi_1'+\eta\varphi_1''-\left(-2F_0'f_4'+F_0f_4''+f_4F_0''\right)$$

$$L_5(\varphi_{14})=\varphi_4'+\eta\varphi_4''-\left(-2F_0'f_1'+F_0f_1''+f_1F_0''\right)$$

$$L_5(\varphi_5)=-\left(1-F_0'^2+F_0F_0''\right)$$

All equations of this system by introducing the new indenpendently variable

$$\eta_1 = \sqrt{\frac{3}{4}}\eta,$$

could be transformed to new forms

$$F_0''' + 2\eta_1 F_0'' = 0$$

$$f_1''' + 2\eta_1 f_1'' - 4f_1' = -\frac{4}{3}(1 - F_0' - \eta_1 F_0'')$$

. . . .

The obtained differential equations are of parabolic type [11]

(9.3) $$y'' + 2xy' - 4\alpha y = 0$$

and for boundary conditions

$$y = y_0, \qquad \text{for } x = 0$$

(9.4) $$y = 0, \qquad \text{for } x = \infty,$$

they have the solutions [11]

(9.5) $$y = y_0\, 2^{\alpha + \frac{1}{2}} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}} e^{-x^2/2} D_{-2\alpha-1}(\sqrt{2}x),$$

respectively, as

(9.6)

$$D_{-2\alpha-1}(\sqrt{2}x) = 2^{\alpha - \frac{1}{2}}\sqrt{\pi}\, e^{x^2/2} g_\alpha(x),$$

where

(9.7) $$g_\alpha(x) = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_x^\infty (\gamma - x)^{2\alpha} e^{-\gamma^2} d\gamma,$$

the Gaussian function of error, while $\Gamma(\alpha+1)$ — gamma function, the solution would be reduced to

(9.8) $$y = y_0\, 2^{2\alpha}\; \Gamma(\alpha+1)\, g_\alpha(x)$$

This means that the solution of the equation (9.3) is to be found in the form

(9.9) $$y = c\, g_\alpha(x),$$

but taking into account that

(9.10) $$g_\alpha(0) = \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(2\alpha+1)},$$

one would find that the solution for conditions (9.4) is given with (9.8).

When $-2\alpha-1=n$ then the function $D_{-2\alpha-1}(\sqrt{2}\,x)$ is reduced to Hermit's polinomial

$$D_{-2\alpha-1}(\sqrt{2}\,x)=2^{\alpha+\frac{1}{2}}\,e^{-x^2/2}\,H_{-2\alpha-1}(x),$$

respectively

$$\text{(9.11)}\quad g_{-\alpha}(x)=\frac{2}{\sqrt{\pi}}\,e^{-x^2}\,H_{2\alpha-1}(x);\; H_{2\alpha-1}(x)=(-1)^{2\alpha-1}\,e^{x^2}\,\frac{d^{2\alpha-1}}{dx^{2\alpha-1}}(e^{-x^2}),$$

while when $-2\alpha-1=-n$ it is reduced to the function of probability

$$D_{-2\alpha-1}(\sqrt{2}\,x)=\frac{\sqrt{\pi}}{\sqrt{2}}\frac{(-1)^{2\alpha}}{(2\alpha)!}\,e^{-x^2/2}\,\frac{d^{2\alpha}}{dx^{2\alpha}}\left\{e^{x^2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^x e^{-\gamma^2}\,d\gamma\right]\right\},$$

respectively

$$\text{(9.12)}\quad g_{\alpha}(x)=(-1)^{2\alpha}\,\frac{1}{2^{\alpha}(2\alpha)!}\,e^{-x^2}\,\frac{d^{2\alpha}}{dx^{2\alpha}}\left\{e^{x^2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^x e^{-\gamma^2}\,d\gamma\right]\right\}.$$

As we have concluded all preparations for solution of the given system (9.2) we shall give also some of its solutions

$$F_0'(\eta_1)=1-g_0(\eta_1);\quad F_0(\eta_1)=\eta_1+g_{1/2}(\eta_1)-\frac{1}{2}\,\frac{1}{\Gamma(3/2)};\quad F_0''(\eta_1)=g_{-1/2}(\eta_1),$$

$$f_1'(\eta_1)=-\frac{4}{3}\,g_1(\eta_1)+\frac{1}{3}\,g_0(\eta_1)-\frac{1}{12}\,g_{-1}(\eta_1),$$

$$f_{11}'(\eta_1)=-\frac{2^4}{3}\,g_2(\eta_1)+\frac{4}{3}\,g_1(\eta_1)-\frac{1}{6}\,g_0(\eta_1)+\frac{1}{108}\,g_{-1}(\eta_1)-\frac{1}{288}\,g_{-2}(\eta_1),$$

$$\text{(9.13)}\quad f_2'(\eta_1)=-\frac{16}{3}\,g_2(\eta_1)+\frac{1}{6}\,g_0(\eta_1)+\frac{1}{18}\,g_{-1}(\eta_1),$$

$$f_{111}'(\eta_1)=-\frac{320}{9}\,g_3(\eta_1)+\frac{80}{9}\,g_{3/2}(\eta_1)+\frac{10}{9}\,g_{1/2}(\eta_1)-\frac{5}{54}\,g_{-1/2}(\eta_1)+$$

$$+\frac{1}{162}\,g_{-3/2}(\eta_1)+\frac{1}{2592}\,g_{-5/2}(\eta_1)+\frac{1}{10368}\,g_{-7/2}(\eta_1),$$

$$f_{12}'(\eta_1)=\frac{1216}{9}\,g_3(\eta_1)-\frac{80}{9}\,g_2(\eta_1)+\frac{2}{9}\,g_1(\eta_1)-\frac{7}{54}\,g_0(\eta_1)+$$

$$+\frac{1}{108}\,g_{-1}(\eta_1)-\frac{1}{216}\,g_{-2}(\eta_1),$$

$$f_3'(\eta_1)=-\frac{128}{3}\,g_3(\eta_1)+\frac{1}{9}\,g_0(\eta_1)-\frac{1}{24}\,g_{-1}(\eta_1),$$

$$\varphi_1'(\eta_1) = -\frac{2}{3}\left(3+\frac{1}{3}\frac{1}{\Gamma^2(^3/_2)}\right) g_1(\eta_1) + \frac{2}{3} g_0(\eta_1) - \frac{1}{9}\frac{1}{\Gamma^2(^3/_2)} g_{-1/2}(\eta_1) +$$

$$+\frac{1}{12} g_{-1}(\eta_1) + \frac{2}{3} g_{1/2}^2(\eta_1) - \frac{2}{3} g_0(\eta_1)\, g_1(\eta_1),$$

$$\varphi_{11}'(\eta_1) = -\frac{64}{3}\left(\frac{1}{3}-\frac{1}{4}\frac{1}{\Gamma^2(^3/_2)}\right) g_2(\eta_1) + \frac{2}{3}\left(3+\frac{1}{3}\frac{1}{\Gamma^2(^3/_2)}\right) g_1(\eta_1) -$$

$$-\frac{4}{27}\frac{1}{\Gamma(^3/_2)} g_{1/2}(\eta_1) - \frac{1}{18}\left(5+\frac{1}{3}\frac{1}{\Gamma^2(^3/_2)}\right) g_0(\eta_1) +$$

$$+\frac{1}{54}\frac{1}{\Gamma(^3/_2)} g_{-1/2}(\eta_1) + \frac{7}{108} g_{-1}(\eta_1) - \frac{1}{108}\frac{1}{\Gamma(^3/_2)} g_{-3/2}(\eta_1) +$$

$$+\frac{1}{144} g_{-2}(\eta_1) - \frac{4}{3} g_1^2(\eta_1) - \frac{2}{3} g_{1/2}^2(\eta_1) - \frac{1}{18} g_0^2(\eta_1) -$$

$$-\frac{4}{3} g_{1/2}(\eta_1)\, g_{3/2}(\eta_1) + \frac{2}{3} g_0(\eta_1)\, g_1(\eta_1) + \frac{1}{9} g_{-1/2}(\eta_1)\, g_{1/2}(\eta_1) -$$

$$-\frac{8}{9} g_0(\eta_1)\, g_2(\eta_1) - \frac{1}{18} g_{-1}(\eta_1)\, g_1(\eta_1),$$

$$\varphi_2'(\eta_1) = -\left(8+\frac{64}{45}\frac{1}{\Gamma^2(^3/_2)}\right) g_2(\eta_1) + \frac{1}{3} g_0(\eta_1) - \frac{1}{15}\frac{1}{\Gamma(^3/_2)} g_{-1/2}(\eta_1) +$$

$$+\frac{1}{18} g_{-1}(\eta_1) + \frac{4}{3} g_1^2(\eta_1) + \frac{2}{3} g_{1/2}^2(\eta_1) - \frac{4}{3} g_{1/2}(\eta_1)\, g_{3/2}(\eta_1) -$$

$$-\frac{2}{3} g_0(\eta_1)\, g_1(\eta_1).$$

It is also easy to obtain the other solutions, but we will not cite them here as they are too long*.

With regard to that we have explained the whole method and given (some) solutions of the system of differential equations (9.2) we can proceed to the consideration of the question of convergence series (2.11). While in Section 2 we have spoken only about the influence of the contour on the goodness of the convergence of this series, here, even if not in completeness, we shall give its proof. The series

$$\mathfrak{F}(x,\eta,\tau) = F(\eta,\tau) + V'(x)\,\Phi(\eta,\tau) + V'^2(x)\,H(\eta,\tau) + V(x)\,V''(x)\,H_a(\eta,\tau) + \cdots,$$

because of (6.3) can be written in the form

$$\mathfrak{F} = \sum_{k=0}^{\infty} \left[F_k\,\tau^k + V'_{k+1}\,\tau^{k+1} + (V'^2\,H_{k+2} + VV''\,H_{a\,k+2})\,\tau^{k+2} + \cdots\right]. \tag{9.14}$$

* But in my thesis there are solutions for 62 equations which correspond to stopping at the second term of series (2.8) and at the fourth degrees of series for F (η, τ) and Φ (η, τ).

For the function $\Omega(t)$ we have earlier supposed that it is bounded in a domain $\mathfrak{D}$, namely

$$\sup_{t\in\mathfrak{D}} |\Omega(t)| = N \tag{9.15}$$

then, if

$$\sup_t \int dt = A,$$

and the function $\tau(t)$ is bounded

$$\sup_{t\in\mathfrak{D}} |\tau(t)| \leqslant \frac{N^2 A}{\nu} \tag{9.16}$$

Let the function $\Omega(t)$ be once differentiable, then the local principal function is bounded

$$\sup_{t\in\mathfrak{D}} |\alpha(\tau)| \leqslant 3. \tag{9.17}$$

Hence it follows that α_k makes bounded a set of numbers as the function f_1, f_{11} f_1, are bounded, what one can see from (9.13), this the functions F_k, Φ_{k+1}, are also bounded

$$\begin{aligned} &F_k \leqslant d_0, \\ &\Phi_{k+1} \leqslant d_1, \\ &H_{k+2} \leqslant d_2, \quad H_{a\,k+2} \leqslant d_{a\,2}, \end{aligned} \tag{9.18}$$

and let be for instance $d_2 \geqslant d_{a2}, \ldots$ Consequently we can make the majorization of series (9.14)

$$\mathfrak{F} \leqslant \sum_{k=0}^{\infty} \tau^k \left[d_0 + V' d_1 \tau + (V'^2 + VV'') d_2 \tau^2 + \cdots\right]. \tag{9.19}$$

The function $V(x)$ is by our supposition quite sufficiently differentiable and its derivatives make a bounded set of functions in an interval J, namely

$$\sup_{x\in J} |V^{(n)}| = M_n. \tag{9.20}$$

Then, if

$$\sup_x \int dx = B,$$

then

$$\sup_{x\in J} |V^{(n-m)}| \leqslant \frac{M_n B^m}{m!} \tag{9.21}$$

Taking into account that $V'^2 > VV'', V'^3 > VV'V'' > VV''', \ldots$ one can make the further majorization of the above series

$$\mathfrak{F} < \sum_{k=0}^{\infty} \tau^k \left[d_0 + \sum_{j=1}^{n} d_j (V'\tau)^j\right], \tag{9.22}$$

respectively because of (9.21)

$$\mathfrak{F} < \sum_{k=0}^{\infty} \tau^k \left[d_0 + \sum_{j=1}^{n} d_j \frac{1}{((n-1)!)^j} (D\,\tau)^j \right] \tag{9.23}$$

where

$$D = M_n B^{n-1}.$$

The first series is convergent for every $\tau > 1$, while at n sufficiently great the second is convergent for every D and τ. That means that the necessary condition for the convergence of series (2.11) is $\tau < 1$, respectively because of (9.16)

$$A < \frac{\nu}{N^2}. \tag{9.24}$$

It remains still open the question of extending the domain of the convergence outside of $\tau < 1$, with respect that it is very difficult and even impossible to obtain the general analytical lawfulness for coefficients $F_k, \Phi_{k+1}, \ldots$ $(k = 0, 1, \ldots)$ in power series and to show that they make diminishing sequences of functions. Therefore, this question should be observed from case to case.

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