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THE ANALYTIC SOLUTIONS OF MIKUSIŃSKI OPERATIONAL DIFFERENTIAL EQUATIONS

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In the present paper K denotes the field of Mikusiński operators [2]. C is the ring of functions $f(t)$, which are defined and continuous in the interval $0 \leqslant t < \infty$. The functions $f(t) \in C$ may have real or complex values. K contains a subring $\overline{C}$ isomorphic to C. The element of K which is isomorphic to $f(t) \in C$ is denoted with f or $\{f(t)\}$. $W(z)$ stands for an operation function ([2] p. 179), and $\{W(z, t)\}$ for a parametric function ([2] p. 179). The limit of sequence of operators, and the derivative $W'(z)$ of an operational function, are defined in ([2] p. 144) and ([2] p. 183).

The operators of J. Mikusiński are very useful for solving the equations containing the operations of translation, integration ($l = \{1\}$), differentiation ($s = \frac{I}{l}$; I identities operator), composition etc.; since those belong to K. However, the theory of operational differential equation has not yet been satisfactory developed.

J. Mikusiński proved the uniqueness of the initial value problem for the equation

$$\sum_{i=0}^{n} a_i W^{(i)}(z) = h(z),$$

where a_i are constant operators, and also the existence in the special case when

$$a_i = \sum_{k=0}^{n_i} b_k s^k, \quad b_k \text{ complex, [1], [3].}$$

B. Stanković proved the existence and the uniqueness of the initial value problem for the equation

$$W'(z) + a(z) W(z) = h(z).$$

$a(z)$, $h(z)$ belonging to certain classes of functions [4], [5].

We consider the operational differential equation of the type

$$\sum_{i=1}^{n} g_i s^{\alpha_i} W^{(n-i)}(z) = 0, \quad g_i \in \overline{C}, \ \alpha_i \text{ real,} \tag{1}$$

which can be reduced to

(2) $$\sum_{i=1}^{n} f_i s^{i\alpha} W^{(n-i)}(z) = 0, \text{ where } f_i = l^{i\alpha-\alpha_i} g_i \in \overline{C},$$

α fixed real number such that $\alpha_i \leqslant i\,\alpha$, for $i=1, 2, \ldots\ldots, n$.

Proposition 1. Let a_i, $b_i \in \overline{C}$, for $i=0, 1, \ldots\ldots, n-1$ $f_o = I$. If $0 < \alpha < 1 + \frac{1}{n}$ then the equation (2) has a unique solution satisfying initial conditions

$$W^{(i)}(z_o) = \frac{a_i}{b_i},\ i=0, 1, \ldots\ldots, n-1.$$

This solution can be represented by an operational power series $\sum_{k=0}^{\infty} e_k s^{\alpha k} z^k$, which converges in every finite interval $[z', z'']$.

The proof of the existence is based on the following three Lemmas:

Lemma 1. Let $u_k \in \overline{C}, k=1, 2, \ldots$. If for each finite interval $[0, T]$, there exists an $M > O$ such that $|u_k(t)| \leqslant \frac{M^k}{k^{\beta k}}$, then the series

(3) $$\sum_{k=0}^{\infty} \frac{1}{(k-i)!} s^{\alpha k} u_k z^{k-i}, \quad \frac{1}{(k-i)!} = 0 \text{ for } k < i,$$

converges for $\alpha < 1 + \beta$, $i=0, 1, \ldots n$, $z \in [z', z'']$

Proof. Let $F(t) = t^{-1}\Phi(0, -\sigma, -t^{-\sigma})$, $0 < \sigma < 1$,

where $\Phi(\nu, \rho, z)$ is the Wright's function [6]. Multiplying the series (3) by F one obtains

(3a) $$\sum_{k=0}^{\infty} \frac{1}{(k-i)!} z^{k-i} u_k l^{k(p-\alpha)+1} s^{kp+1} \{F(t)\}$$

Since $F^{(n)}(0) = 0$, $n=0, 1, \ldots$, and since

$F^{(k)}(t) = t^{-k-1}\Phi(-k, -\sigma, -t^{-\sigma})$, and

$$\left\{\frac{t^{k(p-\alpha)}}{\Gamma_{(kp-k\alpha+1)}}\right\} \{t^{-kp-2}\Phi(-kp-1, -\sigma, -t^{-\sigma}) = t^{-\alpha k-1}\Phi(-\alpha k, -\sigma, -t^{-\sigma}) = F_{\alpha k},$$

from (3) follows

(4) $$\sum_{k=0}^{\infty} \frac{1}{(k-i)!} \{u_k(t)\}\{F_{\alpha k}(t)\} z^{k-i}.$$

Since for $t \in [0, T]$ ($[0, T]$ in the present paper denotes each finite interval.)

$$|t^{-\alpha k-2}\Phi(-k\alpha-1, -\sigma, -t^{-\sigma})| \leqslant \frac{2}{\sigma} \frac{\Gamma\left(\frac{k\alpha+2}{\sigma}\right)}{\left(\cos\frac{\pi\sigma}{2}\right)^{\frac{k\alpha+2}{\sigma}}}$$

for $t \in [0, T]$ there exists a fixed positive number L such that

$$\left| \frac{1}{(k-i)!} u_k(t) z^{k-i} F_{\alpha k}(t) \right| \leqslant k^{-k\left(1+\beta-\frac{\alpha}{\sigma}\right)} L^k.$$

Therefore, for $\alpha < (1+\beta)\sigma < 1+\beta$, the series (4) converges uniform in each finite interval $[0, T]$, which proves Lemma 1.

Lemma 2. Let $u_k \in \overline{C}$, for $k = 1, 2, \ldots$, $0 < \alpha < 1+\beta$, and let for $t \in [0, T]$ exist an $L > 0$ such that

$$|u_k(t)| \leqslant \frac{L^k}{k^{\beta k}}.$$ Then the series

(5) $W(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \{u_k(t)\} s^{\alpha k} z^k$ may be termwise differentiated in $z \in [z', z'']$ arbitrary many times.

Proof. Multiplying the series (5) by F, one obtains the parametric function

$$F\,W(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \{u_k(t)\} \{F_{\alpha k}(t)\} z^k. \tag{5a}$$

Then the derivative of series (5) is

$$W'(z) = \frac{I}{F} \sum_{k=0}^{\infty} \frac{1}{(k-1)!} \{u_k(t)\} F_{\alpha k} z^{k-1} = \sum_{k=0}^{\infty} \frac{1}{(k-1)!} u_k s^{\alpha k} z^{k-1}. \tag{6}$$

The convergence of series (6) follows from Lemma 1.

Lemma 3. Let $f_i(t) \in C$, for $i = 1, 2, \ldots n$, and $u_j(t) \in C$, $j = 1, 2, \ldots, n-1$.

Let $u_{n+k}(t) = \sum_{i=1}^{n} \int_0^t f_i(t-\tau) u_{n+k-i}(\tau) d\tau$, for

$k = 0, 1, \ldots$, then for $t \in [0, T]$ exists an $L > 0$ such that

$|u_{n+k}(t)| \leqslant \frac{L^k}{k^{\frac{1}{n}k}}$ for $k = o, 1, \ldots$.

Proof. Let $M = \max(|f_i(t)|, |u_j(t)|)$ for $t \in [0, T]$, $i = 1, 2, \ldots, n$, and $j = 0, 1, \ldots, n-1$.

By method of total induction we can show that

$$|u_{n+k}(t)| \leqslant 2^k n M \sum_{i=1+\left[\frac{k}{n}\right]}^{k+1} \frac{(Mt)^i}{i!}; \tag{a}$$

for $k = 0$ we have

$$|u_n(t)| \leqslant \sum_{i=1}^{n} \int_0^t |f_i(t-\tau) u_{n-i}(\tau)| d\tau \leqslant 2^{\circ} n M^2 t.$$

Now suppose that

$$|u_{n+k}(t)| \leqslant 2^k n M \sum_{i=1+\left[\frac{k}{n}\right]}^{k+1} \frac{(Mt)^i}{i!} \text{ for } k=k_o, k_o+1, \ldots, k_o+n-1.$$

$$\text{then } |u_{2n+k_o}(t)| \leqslant \sum_{j=0}^{n} \left| \int_0^t f_j(t-\tau)\, u_{2n+k_o-j}(\tau)\, d\tau \right| \leqslant$$

$$M \sum_{j=0}^{n} \sum_{i=1+\left[\frac{n+k_o-j}{n}\right]}^{n+k_o-j+1} 2^{n+k_o-j} n \frac{(Mt)^{i+1}}{(i+1)!} = n M \sum_{i=1+\left[\frac{n+k_o}{n}\right]}^{n+k_o+1} \frac{(Mt)^i}{i!} \left(\sum_{j=i-2}^{n+k_o-1} 2^j \right) \leqslant$$

$$\leqslant 2^{n+k_o} nM \sum_{i=1+\left[\frac{n+k_o}{n}\right]}^{n+k_o+1} \frac{(Mt)^i}{i!}$$

From (a) immediately follows Lemma 3.

Proof of the proposition 1. Let be $z_o=0$ and $g=\dfrac{I}{b_o b_1 \ldots b_{n-1}}$ we show that in

$$W(z)=g \sum_{k=1}^{\infty} \frac{1}{k!} u_k s^{\alpha k} z^k \tag{7}$$

the u_k can be determined so, that (7) satisfies formally the equation (2) and the initial value $W^{(i)}(0)=\dfrac{a_i}{b_i}$, $i=0, 1, \ldots, n-1$. Put $V_k=u_k F_{\alpha k}$, then (7) becomes

$FW(z)=g \sum_{k=0}^{\infty} \frac{1}{k!} V_k z^k$, and by differentiating the above formula and using Lemma 2 we have

$$W^{(i)}(z)=\frac{g}{F} \sum_{k=0}^{\infty} \frac{1}{(k-i)!} V_k z^{k-i}$$

By substituing the above formulas into (2) we get

$$\sum_{i=0}^{n} f_i s^{i\alpha} \sum_{k=o}^{\infty} \frac{1}{(k-n+1)!} V_k z^{k-n+i} \tag{8}$$

or

$$\sum_{k=0}^{\infty} \sum_{i=0}^{n} f_i s^{(i-n-2)\alpha} \frac{1}{k!} V_{k+n-i} z^k = 0.$$

Since

$$f_i s^{\alpha(i-n-2)} \frac{1}{k!} V_{k+n-i} \in \overline{C}$$

we have

$$\sum_{i=0}^{n} f_i u_{k+n-i} = 0 \quad k = 0, 1, \ldots \tag{9}$$

From (9) we can determine u_k successively:

$u_k = -\sum_{i=1}^{n} f_i u_{k-i} \in \bar{C}$, for $k = n, n+1, \ldots$ and for $k = 0, 1, \ldots n-1$ u_k is determined by initial value; $u_k = l^{\alpha k} a_k b_o b_1 \ldots b_{k-1} b_{k+1} \ldots b_{n-1} \in \bar{C}$ which was to be proved. The convergence of $W(z), z \in [z', z'']$ follows from Lemma 1 and Lemma 3.

The uniqueness part of the proposition 1 follows from a result of S. Drobot and J. Mikusiński [1].

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