PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 6 (20), 1966. p. 165 — 170

IMBEDDING OF ORDERED SETS IN MINIMAL LATTICES

Đuro Kurepa

(Communicated 4. February 1966)

1. Let

(1) $$(O, <)$$

be any ordered set; it is known that the set (1) is imbeddable into a complete lattice $L(O)$ (Mac Neille; cf. Birkhoff, p. 58; Szász p. 73). In general case, the cardinality kO of O is less than the cardinality $kL(O)$ of $L(O)$; now the question arises as whether the imbedding of $(O, <)$ into a lattice $(O_R, <_R)$ is feasible *under the condition that O and O_R be of same cardinality.*

We are going to prove that for any infinite O the answer is by affirmative; for any finite ordered set $(0, <)$ which is not a lattice the answer is by negative.

2. Let us consider the following problem:

Problem. *Let n be any cardinal number; consider the minimal number m (n) such that every ordered set of a cardinality n be isomorph to a subset of a lattice of cardinality* $\leqslant m(n)$.

The existence of $m(n)$ for any given n is obvious: the question is to determine $m(n)$ as function of n. E. g. $m(0)=0, m(1)=1$ (the empty set as well as every one-point set are considered as lattices); $m(2)=4$.

3. Theorem. *For every cardinal number $n > 1$ one has*

(1) $$m(n) \leqslant 2^n.$$

If $n \geqslant \aleph_0$, then $m(n)=n$.

In particular, every infinite ordered set $(0, <)$ is imbeddable into a lattice $(O_R, <_R)$ of the cardinality kO.

4 Proof. 4. 1. The first step in the transition $(O, <) \rightarrow (O_R, <_R)$ consists to adjoin to O: a first member, 0, provided it is not present in $(O, <)$ and to adjoin a last member, 1, provided it is not present in $(O, <)$.

The relation (1) is an immediate consequence of the forming of $L(O)$ by means of subsets of O, all these subsets forming the partitive set PO of cardinality 2^n.

4.2 As to the existence of $(O_R, <_R)$, at first we define O_a as the family of all the sets of the form

$$0\ 1X, (X \subseteq O, kX < \aleph_0)$$

$$\text{where } 1\, X = \{y; y \in O, X < y\}$$
$$0\, X = \{x; x \in O, x < X\}$$
$$0\, 1\, X = 0(1\, X).$$

Then $(O_a; \subseteq)$ is a lattice.

4.3 If for every $x \in O$ we substitute x for $O\ (., x)$ and $\leqslant_R$ for $\subseteq$, then O_a yields a set, say O_R and $O_R \supseteq O$; the set $(O, <)$ is imbedded in $(O_R, <_R)$.

4.4 *l-extension of* $(O, <)$. Moreover, the set $(O_R, <_R)$ is an *l-extension of* $(O, <)$ in the sense that not only $O_R \supseteq O$ and

$$a \leqslant b \text{ in } O \Rightarrow a \leqslant_R b \text{ in } O_R$$

but also that for $\{a, b\} \subseteq O$ one has

$$\inf_0 \{a, b\} \in O \Rightarrow \inf_0 \{a, b\} = \inf_{O_R} \{a, b\},$$
$$\sup_0 \{a, b\} \in O \Rightarrow \sup_0 \{a, b\} = \sup_{O_R} \{a, b\}.$$

4.5. The cardinality of O_a is such that

$$k\, 0 \leqslant k\, 0_a \leqslant k\, 0 + k\, 0^2 + k\, 0^3 + \dots$$

4.6. Consequently, assuming the axiom of choice one has $k\, O^r = k\, O$ for every natural integer r and every infinite O. Therefore $k\, O_a = k\, 0$ and also $k\, 0_R = k\, 0$ because $k\, 0_a = k\, 0_R$.

4.7. For another proof of the theorem 3 cf. section 8.

5. *Extension of the validity of relations inf* $\{a, b\} \in O$ *and sup* $\{a, b\} \in O$.

5.1 Sometimes it is interesting to imbed $(O, <)$ in a (minimal) l-extension (M, ρ) of $(O, <)$ in such a way that for a given

$$\text{set } E \subseteq \binom{0}{2} = \{X; X \subseteq O, k\, X = 2\}, \text{ one has}$$

$$x \in E \Rightarrow \inf_M x \in M \text{ or}$$
$$x \in E \Rightarrow \sup_M x \in M \text{ or both.}$$

In general, the ordered set (M, ρ) is not a lattice. The simplest case is that E consists of a single 2-point-subset of O.

5.2. We are going to indicate a construction of $M = M(E)$ leaving aside the question whether the construction of $M(E)$ is as economical as possible in the sense to introduce in $M(E)$ as many comparable elements as possible.

5.3. Lemma. *Let* $(O, <)$ *be an ordered set; let* $\{a, b\} \subseteq O$ *and*

$$i = \inf \{a, b\} \in 0;$$

let (M, ρ) *be an ordered set extending* $(0, <)$ *by adjoining to the set O a single member* $x \notin O$.

Then we have the following equivalence $(1) \Leftrightarrow (2)$ *where*

(1) $\quad i \,\|_\rho\, x$,[1)] $x\, \rho\, \{a, b\}$

[1)] $i \,\|_\rho\, x$ means that neither $i\, \rho\, x$ nor $x\, \rho\, i$ (i. e. that i, x are ρ incomparable

(2) $inf_M\{a,b\} \notin M$.

The implication (1) ⇒ (2) is obvious, because (1) and the relation $i \in O$ imply that the set of all predecessors of a, b in (M, ρ) consists of x and of O[i,.) and has 2 initial points i, x.

Conversely, (2) ⇒ (1). At first, from (2) we infer that

(3) $M(.,a] \cap M(.,b] = O(.,a] \cap O(.,b] \cup \{x\}$ and thus $x \rho \{a,b\}$.

One has neither $x \rho i$ nor $i \rho x$ because these relations would imply that $\inf_M \{a,b\}$ equals x and i respectively, in contradiction with (2).

The dual of 5.3 reads as follows.

5.4. *Lemma. Let* $(O, <)$ *be an ordered set; let* (M, ρ) *be an order extension of* $(0, <)$ *obtained from* $(0, <)$ *by adjoining a single point* y; *if* $\{a\ b\} \subseteq O$ *and* $s = sup_0\{a,b\} \in O$, *then the following conditions* (4), (5) *are logically equivalent*:

(4) $s \|_\rho y, \ \{a,b\} \rho y$

(5) $\sup_M\{a,b\} \notin M$.

5.5. Lemma. Let $(O, <)$ be any ordered set; if $u, v \in O$ and if $\inf_0 \{u,v\}$ does not exist, let $x = x(u,v)$ be an object which is neither a member nor a part of O: let locate the object x immediately before a and before b and immediately after the set

(1) $O(.,u] \cap O(.,v]$; in particular $\bar{u} = x = \bar{v}$;

for any other point t of O we consider t, x to be incomparable.

If the set (1) is empty we define x to follow to every point of $(0, <)$. The ordered set $(M = O \cup \{x\}; <')$ so obtained is an extension of the given ordered set $(O, <)$ leaving invariant supremum as well as infimum of any 2-point-subset $\{a,b\}$ of O; i. e. if $\{a,b\} \subseteq O$, then

(2) $i = \inf_O\{a,b\} \in O \Rightarrow \inf_O\{a,b\} = \inf_M\{a,b\}$:

(3) $s = \sup_O\{a,b\} \in O \Rightarrow \sup_M\{a,b\} = \sup_M\{a,b\}$.

Let us prove the implication (2).

First $\inf_M\{a,b\}$ *exists and is a member* t *of* M; in opposite case, the implication (2) ⇒ (1) in the lemma 5.3. would yield

(4) $x \rho \{a,b\}$ and $i \|_\rho x$.

In particular, $x \rho \{a,b\}$ implies $u \leqslant \{a,b\}, v \leqslant \{a,b\}$ and from here, by the definition of $i = \inf_O \{a,b\}$, one would have $\{u,v\} < i$, and from here $x \rho i$, contradicting the second relation of (4).

Hence $t \in O$. Therefore $i \rho t$. We say that $t \in O$ i. e. $i = t$ and that (2) holds. In opposite case, one would have $t = x$, thus $i \rho x$, and $i \leqslant \{u,v\}$. Now $x (= t) = \inf_M\{a,b\}$; therefore $x \rho \{a,b\}$ and $u \leqslant \{a,b\}, v \leqslant \{a,b\}$; thus $\{u,v\} \leqslant i$ and $x \rho i$, contradicting $i \rho x, i \neq x$.

Analogously, (3) is holding.

First, $\sup_M\{a, b\}$ exists and is a member z of M. Otherwise one would apply the implication (5) $\Rightarrow$ (4) in the lemma 5.4 and consequently one would have

$$(5) \qquad s \,\|_\rho\, x, \qquad \{a, b\}\, \rho\, x.$$

The last relation implies $\{a, b\} \leqslant u, \{a, b\} \leqslant v$, hence by the definition of $s = \sup_O\{a, b\}$, $s \leqslant \{u, v\}$ and therefore $s \rho x$, contradicting (5).

On the other hand, $z \in M \Rightarrow z \rho s$. If moreover $z \in O$, then necessarily $s = z$ and the requested implication (3) is proved. Now, suppose by contradiction that $z \in M \setminus O$, i. e. $z = x$. Then $x \rho s$, $x \neq s$ and $\{u, v\} < s$. Further, $x = z = \sup_M\{a, b\}$ implies $\{a, b\} \rho x$; from here by the definition of ρ we have $\{a, b\} \leqslant u, \{a, b\} \leqslant v$ and therefore also $s \leqslant \{u, v\}$, hence $s \leqslant x$ contradicting the assumption $x \rho s$, $x \neq s$.

Since the last sentence in the lemma is obvious, the proof of the lemma is completed.

The dual of the lemma 5.5. reads as follows.

5.6. Lemma. Let $(O, <)$ be any ordered set; if $\{u, v\} \subseteq O$ and $\sup_O\{u, v\} \notin O$, then adjoining to O an object $y = y(u, v)$ which is neither a member nor a part of 0 and defining in $M = O \cup \{y\}$ the extension ρ of $\leqslant$ in such a way that $u^+ = y = v^+$ and that every member of the set $O[u,.) \cap O[v,.)$ precedes immediately to y, while else y is incomparable to every other point of O then for any $\{a, b\} \subseteq O$ one has the implications:

$$\inf_O\{a, b\} \in O \Rightarrow \inf_{(O,<)}\{a, b\} = \inf_{(M\rho)}\{a, b\},$$

$$\sup_O\{a, b\} \in O \Rightarrow \sup_{(O,\leqslant)}\{a, b\} = \sup_{(M,\rho)}\{a, b\}$$

$$y = \sup_{(M,\rho)}\{u, v\}.$$

The proof runs dually to that of the Lemma 5.5.

6. Theorem. *Let $(O, \leqslant)$ be any ordered set and $E \leqslant \binom{O}{2}$ any set of 2-point-subsets of O; there exists an ordered set $(O(E), \leqslant(E))$ extending the ordered set $(O, \leqslant)$ such that $\{u, v\} \in E \Rightarrow \inf_{O(E)}\{u, v\} \in O(E)$: moreover, if $k\,0 < \aleph_o$, then $k\,0 = k\,0\,(E)$.*

And dually for the supremum for every $\{x, y\} \in F$ where $F \subseteq \binom{O}{2}$ is given.

Proof. Let

$$a_0, a_1, \ldots a_\varphi, \ldots (\varphi < \Psi)$$

be any normal well-order of E; for every $\varphi < \Psi$ we have

$$a_\varphi = \{a_{\varphi 0}, a_{\varphi 1}\} \subseteq O, \qquad a_{\varphi 0} \neq a_{\varphi 1}.$$

6.1. We define

$i\,E = \{(i, x); x \in E\}$; i is the first character of the word infimum; s is the initial character of supremum; obviously, the sets $E, i\,E$, are disjoint.

6.2. Let us consider the following ordered sets

$$(0_{1\varphi}, \leqslant_{\varphi 1})\ (\varphi < \Psi)$$

extending $(0, \leqslant)$.

If $\inf a_0 \in (0, \leqslant)$ we put $(0_{10}, \leqslant_{10}) = (0, \leqslant)$; if

$\inf a_0 \notin (0, \leqslant)$, then $0_{10} = 0 \cup \{i a_0\}$ and we define $\leqslant_1$ by

intercalling in $(0, \leqslant)$ the element $(i a_0)$ between a_0 and the set

(4) $\qquad 0(., a_{00}] \cap O(., a_{01}]$.

By the lemma 5.5, in the set $(0_{10}, \leqslant_{10})$ the infimum of $a_0 = \{a_{00}, a_{01}\}$ exists: if sup $\{x, y\}$ exists in $(0, <)$ so also in $(0_{10}, \leqslant_{10})$ and is the same.

6.3. One defines $(0_{11}, \leqslant_{11})$ substituting in the foregoing consideration 5.5, 5.6

$$0 \to 0_{10}, \quad \leqslant \to \leqslant_{10}, \quad a_0 \to a_1; \text{ if}$$

$0 < \gamma < \Psi$ and if the set $(0_{1\beta}, \leqslant_{1\beta})$ is defined for every $\beta < \gamma$, we define also $(0_{1\gamma}, \leqslant_{1\gamma})$ as previously on substituting

$$O \to \bigcup_{\beta<\gamma} O_{1\beta}, \quad < \to \bigcup_{\beta<\gamma} <_{1\gamma}, \quad a_0 \to a_\beta,$$

By definition

$$x \, (\bigcup_{\beta<\gamma} \leqslant_{1\beta}) \, y \text{ means } x \leqslant_{1\beta} y \text{ for at least one } \beta < \gamma.$$

6.4. The ordered set $(0_{1\varphi}, \leqslant_{1\varphi})$ being defined for every $\varphi < \Psi$ we define

$$g(O, <) = (0_1, \leqslant_1) \underset{=\!=}{\text{def}} (\bigcup_{\varphi<\psi} O_{1\varphi}, \sup_{\varphi<\psi} \leqslant_{1\varphi}).$$

6.5. The ordered set $(0_1, \leqslant_1)$ is an extension of $(0, \leqslant)$ and one sees readily that for $\{x, y\} \in E$ one has

$$\inf \{x, y\} \in (0_1, \leqslant_1);$$

moreover, if $x, y \in O$ have its infimum in $(0, \leqslant)$ so do they the same in $(0_1, \leqslant_1)$ with the same value.

6.6. One has $k\,0_1 = k\,0$ because $k\,0 \leqslant k\,0_1 \leqslant k\,0 \; k\,i\,E \leqslant$

$$k\,0 + k\,0^2 = k\,0.$$

The proof for the dual form runs analogously on considering instead of the set iE the set $sE = (sx), x \in F$; s is the initial of the word supremum.

Of course, the sets 0, iE, sE are pairwise disjoint.

7. Theorem. Let $(0, \leqslant)$ be any ordered set and $E \subseteq \binom{0}{2}$, $F \subseteq \binom{0}{2}$; there exists an l-order extension $(M, \leqslant_R)$ of $(0, \leqslant)$ such that

$$e \in E \Rightarrow \inf_M e \in (M, \leqslant_R) \text{ and } f \in F \Rightarrow \sup_M f \in (M, \leqslant_R).$$

The theorem 7 is an easy consequence of the theorem 6 for $E = F = \binom{0}{2}$.

8. *Another proof of the theorem* 3.

Put in the foregoing proof $E = \binom{0}{2} = F$.

Then putting $g(O, \leqslant) = g_1(O, <)$ and defining

$$g(g_r(O, \leqslant)) = g_{1+r}(O, \leqslant) = (0_{1+r}, \leqslant_{1+r})$$

the requested ordered set $(0_R, \leqslant_R)$ is defined in the following way:

$$0_R = \bigcup_{r<r} O, \quad \leqslant_R = \bigcup_r \leqslant_r, \qquad (0 < r < \omega).$$

One proves readily that the ordered set $(O_R, \leqslant_R)$ is a lattice of cardinality $k\,0$ and that $(0_R, \leqslant_R)$ is an l-extension of $(O, \leqslant)$.

BIBLIOGRAPHY

Birkhoff, G., *Lattice Theory*, Amer. Math. Soc., New York, 1948_2 14+284.
Szász, Gábor, *Introduction to Lattice Theory*, Budapest, 1963, 230.