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ON THE DIRICHLET'S PROBLEM FOR THE NAVIER-STOKES EQUATIONS ON A RIEMANNIAN MANIFOLD

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In this paper are considered the existence and the uniqueness of solutions of the Navier-Stokes equations on a n-dimensional Riemannian manifold $\mathfrak{R}$, assuming on $\mathfrak{R}$ the existence of classical Navier-Stokes equations. The assumed classical form of the Navier-Stokes equations may be regarded as a first approximation of the exact Navier-Stokes equations for a Riemannian manifold. It may be expected that the constitutive equations for a viscous fluid on $\mathfrak{R}$ are different from the stress-rate of strain relations in the Euclidean space E_3. The reason for the investigation of solutions in a Riemannian space is that in applications, especially in aeronautics and in hydraulic engineering there are profiles and stream spaces which have not Euclidean but Riemannian metric.

In Chapter 1 is formulated the problem for consideration. Chapter 2 is dedicated to the construction of a fundamental solution for the elliptic equation using Hadamard's idea for the construction of an elementary solution for the establishing of the existence of a parametrix. In Chapter 3 is given the solution of linearized, in the sense of neglected inertial terms, Navier-Stokes equations, with detailed consideration of the question of the existence and the uniqueness of solutions. The solutions are given in the form of potentials of distributions. The influence of the number of dimensions of $\mathfrak{R}$ on the question of the existence of solutions is of interest.

This paper is dedicated to the steady case of the Navier-Stokes equations, while the unsteady one will appear elsewhere.

Contents

1. Introduction

Let $\mathfrak{D}$ be a connected domain with the boundary $\mathfrak{B}$ of an orientable, n-dimensional C^∞ Riemannian manifold $\mathfrak{R}$ with the metric $ds^2 = g_{ij}\, dx^i\, dx^j$, where $g_{ij} = g_{ij}\ (x^1, \ldots\ldots, x^n)$ is the metric tensor, and let $\mathfrak{D}_R$ be a region outside of $\mathfrak{B}$, namely the complement, that represents a continuum of $\mathfrak{D}$ to a sphere S of the infinitely large radius R. Suppose that on $\mathfrak{D}$, respectively on $\mathfrak{D}_R$ exist the following partial equations, given in the system of local coordinates

$$\rho \frac{dv^i}{dt} = \rho\, F^i + t^{ij}{}_{,j}\,, \tag{1.1}$$

$$v^i{}_{,i} = 0\,, \tag{1.2}$$

where

, : covariant derivative with respect to the coefficients of connection $\Gamma^i_{jk} = \left\{ {i \atop jk} \right\}$;

ρ: constant density;

t^{ij}: stress tensor, $t^{ij} = -pg^{ij} + 2\,\mu\, D^{ij}$;

D^{ij}: deformation tensor $D^{ij} = \dfrac{1}{2}\,(v^{i,j} + v^{j,i})$;

μ: dinamical coefficient of viscosity;

p: hydrostatics pressure;

F^i: extraneous force field.

All magnitudes are time independent, and indices take the values $1, \ldots, n$.

A vector field $v^i(x)$, $x = (x^1, \ldots\ldots, x^n)$, which for any given scalar field $p(x)$ satisfies the equations (1.1) and (1.2), will be said that defines a velocity field of a viscous fluid in $\mathfrak{D}$ and $\mathfrak{D}_R$ respectively.

Aim of this paper is; to find unique solutions of the above equations, $v^i(x)$ and $p(x)$ from $C^\infty(\mathfrak{D})$ for any $F^i(x) \in C^\infty(\mathfrak{D})$ and a given vector field f^i on $\mathfrak{B}$

$$(v^i)_{\mathfrak{B}} = f^i \in C^\infty(\mathfrak{B}), \tag{1.3}$$

and unique solutions $v^i(x)$ and $p(x)$ from $C^\infty(\mathfrak{D}_R)$ for any $F^i(x) \in C^\infty(\mathfrak{D}_R)$ and a condition of vanishing of these solutions on the surface $\mathfrak{B}_R$ of an infinitely large sphere, namely

$$(v^i)_{\mathfrak{B}_R} \to 0\,,\ (p)_{\mathfrak{B}_R} \to 0\,, \qquad \text{when } R \to \infty\,. \tag{1.4}$$

To the above conditions ought to be subjoined also a condition that there is no streaming across $\mathfrak{B}$, i. e.

$$\int_{\mathfrak{B}} f^i\, d\mathfrak{B}_i = 0\,. \tag{1.5}$$

We remark that on these problems have been working many authors, but only in the case when $\mathfrak{R}$ is an Euclidean two or three dimensional space. Some of them are: Finn [2], Ladyzhenskaia [11], Leray [13], Lichtenstein [15], Odquist [18] and others. For further informations see [11].

2. The fundamental solution of the elliptic equation on $\mathfrak{R}$

2. 1 *Preliminary investigations*

We consider the differential operator

$$A = b^{ij}(x) \frac{\partial}{\partial x^i \partial x^j} + a^i(x) \frac{\partial}{\partial x^i} + c(x), \tag{2.1.1}$$

on an open set $\mathfrak{G} \subset \mathfrak{R}$. Here $b^{ij}(x)$ is the contravariant tensor such that the quadratic form is positively definite i.e., $b^{ij}(x)\xi_i\xi_j > 0$ for $\sum_i (\xi^i)^2 > 0$, and $a^i(x)$ changes, by a coordinate transformation $x \to \bar{x}$, as follows

$$\bar{a}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} a^j(x) + \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} b^{jk}(x). \tag{2.1.2}$$

These transformation rules for the coefficients are connected with the fact that A is independent of the local coordinates $(x^1, \ldots, x^n)$. For the sake of simplicity, we assume that the coefficients $b^{ij}(x)$, $a^i(x)$ and $c(x)$ belong to $C^\infty(\mathfrak{G})$.

A typical representative of the above operator is the Laplace's operator $\Delta = g^{ij}\nabla_i\nabla_j$, where g^{ij} is the metric tensor, and with ∇ is denoted the covariant differential with respect to the connection $\left\{ {k \atop mn} \right\}$, in the fixed system of local coordinates. It is easy to verify that this operator has the form

$$\Delta = g^{ij}\partial_i\partial_j + A^i\partial_i + B, \tag{2.1.3}$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$, with matrixes as its coefficients. Let us apply this operator to any mixed type tensor Q_j^i, whose matrix of the rank n we shall denote with Q:

$$\Delta Q = g^{ij}\partial_i\partial_j Q + A^i\partial_i Q + BQ. \tag{2.1.4}$$

For the sake of simplicity, we assume further that g^{ij}, A^i and B are infinitely differentiable functions of the local coordinates $(x^1, \ldots, x^n)$.

Let us construct, under a certain hypothesis, which will be satisfied if $\mathfrak{G}$ is compact, the fundamental solution-fundamental tensor of the equation (2.1.4)

$$F(x, \xi), \qquad (x, \xi \in \mathfrak{G}), \tag{2.1.5}$$

with the following properties:

i)

$$\Delta_x F(x, \xi) = 0, \quad \Delta_\xi F(x, \xi) = 0, \tag{2.1.6}$$

where $\xi = (\xi^1, \ldots, \xi^n)$ is a parameter point.

ii) If, according to the metric $ds^2 = g_{ij}\,dx^i\,dx^j$.

$$\Gamma(x, \xi) = s^2(x, \xi)$$

be the square of the geodesic distance between points $x, \xi \in \mathfrak{G}$. Then, if $x \to \xi$, respectively $\Gamma(x, \xi) \to 0$, the fundamental tensor $F(x, \xi)$ exhibits the principal singularity characterized by the following representation

(2.1.7) $$U\Gamma^{-m}+V\log\Gamma+W,$$

where U, V and W are supposed to be regular functions of x in a neighbourhood of ξ, with $U_j^i(\xi,\xi)\neq 0$, and where the exponent m should have the specific value; $m=\frac{n}{2}-1$.

iii) We have

(2.1.8) $F(x,\xi)$ is bounded in $\xi(x)$ for fixed $x(\xi)$,

(2.1.9) $\int_{\mathfrak{S}}|F(x,\xi)|\,d\xi$, where $d\xi=(g)^{1/2}d\xi^1\cdots d\xi^n$ and $g=\det(g_{ij}(x))$, is bounded in x

But, for forming a fundamental solution of the before denoted equation we need a parametrix. Therefore, at first we shall establish the existence of it.

2.2 *The parametrix*

If $\Gamma(x,\xi)$ is the square of the geodesic distance between two neighbour points $x,\xi\in\mathfrak{S}$, then we have the following

Lemma. *For the equation (2.1.4) we may construct the parametrix**

(2.2.1) $$\overset{*}{P}{}_j^i(x,\xi)=\begin{cases}\dfrac{1}{(n-2)\tau_n}\Gamma^{-m}(x,\xi)\cdot U_j^i(x,\xi)+\log\Gamma(x,\xi)\cdot V_j^i(x,\xi)+ & (n>2)\\ \qquad +W_j^i(x,\xi) & \\ -\dfrac{1}{4\pi}\log\Gamma(x,\xi)\cdot U_j^i(x,\xi)+V_j^i(x,\xi), & (n=2),\end{cases}$$

such that

(2.2.2) *$U_j^i(x,\xi)$, $V_j^i(x,\xi)$ and $W_j^i(x,\xi)$ are infinitely differentiable tensor functions of x in the vicinity of ξ, with $U_j^i(\xi,\xi)=\delta_j^i$,*

(2.2.3) *$\Delta_x\overset{*}{P}{}_j^i(x,\xi)$ gives an infinitely differentiable tensor function of x in the vicinity of ξ,*

whereby τ_n denotes the surface area of the n-dimensional unit sphere, and $m=\frac{n}{2}-1$.

Proof. To prove the existence of the parametrix, we shall apply first of all the operator Δ to the product of $l(\Gamma)$ and of a mixed type tensor $Y_j^i(x,\xi)$ considering $l(\Gamma(x,\xi))\,Y_j^i(x,\xi)$ as a function of x. It follows

$$\Delta\, l\, Y_j^i=l\,\Delta\, Y_j^i+2\,g^{mn}\,\nabla_m l\cdot\nabla_n Y_j^i+Y_j^i\,\Delta\, l,$$

respectively, if we take into account that

$$\nabla_m l=l'\,\Gamma_m,$$

$$\nabla_m\nabla_n l=l''\,\Gamma_m\,\Gamma_n+l'\,\nabla_m\Gamma_n,$$

* To this purpose we follow the Hadamard's idea, see [5].

whereby the symbol $\Gamma_i = \partial_i \Gamma$ is introduced, and that

$$\nabla_n Y^i_j = \partial_n Y^i_j + \Gamma^i_{ns} Y^s_j - \Gamma^s_{nj} Y^i_s ,$$

we obtain

(2.2.4) $\Delta l Y^i_j = l \Delta Y^i_j + l' [\Delta \Gamma \cdot Y^i_j + 2 g^{mn} \Gamma_m (\partial_n Y^i_j + \Gamma^i_{ns} Y^s_j - \Gamma^s_{nj} Y^i_s)] + g^{mn} \Gamma_m \Gamma_n l'' Y^i_j$

Let us observe now any point z on the geodesic joining the points x and ξ as a function of $s = s(\xi, z)$. We have then the known identities, see [20]. Let be

(2.2.5) $$L(z, \dot{z}) = g_{ij}(z) \dot{z}^i \dot{z}^j = e , \qquad \text{where } \dot{z}^i = \frac{dz^i}{ds} .$$

e is an indicator of geodesicity. In our case, because the metric is positive definite, $e = +1$. We have further

$$\Gamma_i(x, \xi) = s(x, \xi) \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} = 2 s(x, \xi) g_{ij}(x) \dot{x}^j ,$$

and from here we obtain the important identity

(2.2.6) $$g^{ik}(x) \Gamma_i(x, \xi) \Gamma_k(x, \xi) = 4 \Gamma(x, \xi) .$$

We introduce the normal coordinates y^i of the point x around ξ. Then

$$y^i = \left(\frac{dx^i}{ds}\right)_\xi s = a^i s ,$$

and in that case the square of the geodesic distance obtains the form

(2.2.7) $$\Gamma(x, \xi) = g_{ij}(x) y^i y^j$$

Then, with (2.2.5) we have the important relation

(2.2.8) $$g_{ij} \Gamma_i \partial_j = 2 s \frac{\partial}{\partial s} .$$

By taking that

$$\Delta \Gamma = 2n + x^k \frac{\partial \log g}{\partial x^k} , \qquad g = \det(g_{ij}(x)) ,$$

it is easy to verify that

(2.2.9) $$\Delta \Gamma \to 2n \text{ when } x \to \xi .$$

Let us write now (2.2.4) in the matrix form and introduce the symbol

(2.2.10) $$2n - \Delta \Gamma - A^i \Gamma_i = 4H ,$$

where A^i are the matrix coefficients, which contain in themselves the coefficients of connection $\left\{ {s \atop mn} \right\}$, then taking the expressions (2.2.6), (2.2.8) and (2.2.9) the equation obtained in such a way becomes

$$(2.2.11) \qquad \Delta\, l\, Y = l\, \Delta\, Y + l' \left\{ 2\,n - 4\,H + 4\,s \frac{\partial}{\partial\, s} \right\} Y + 4 \Gamma\, l''\, Y .$$

This differential equation can be made independent explicitly of H defining a new operator

$$(2.2.12) \qquad \overset{M}{\Delta} = M^{-1}\, \Delta\, M,$$

where the matrix M is defined in the following way

$$(2.2.13) \qquad M = \exp \left(\int \frac{H}{s}\, ds \right),$$

and satisfies the following differential equation

$$s \frac{\partial M}{\partial\, s} = HM .$$

In this way we can write the equation (2.2.11) in the new form

$$(2.2.14) \qquad \overset{M}{\Delta}\, l\, Y = l\, \overset{M}{\Delta}\, Y + l' \left\{ 2\,n + 4\,s \frac{\partial}{\partial\, s} \right\} Y + 4\, \Gamma\, l''\, Y .$$

Now, to prove the lemma, we suppose the regular functions $U\,(x, \xi)$ and $V\,(x, \xi)$ in the form of formal power series

$$(2.2.15) \qquad \begin{aligned} U &= M \sum_{\tau=0}^{\nu} U_\tau\, \Gamma^\tau, \\ V &= M \sum_{\tau=1}^{\mu} V_\tau\, \Gamma^\tau, \end{aligned}$$

where the index τ does not denote the tensor character of magnitudes, but by it one does the summing, and $U_0 = 1$.

If now the operator (2.2.12) is applied to (2.2.1) in the way given by (2.2.14) taking (2.2.15) we obtain

$$\overset{M}{\Delta} M^{-1} \overset{*}{P} = \Gamma^{-m} \sum_{\tau=1}^{\nu} \Gamma^{\tau-1} \left\{ 4\,(\tau - m) \left(\tau + s \frac{\partial}{\partial\, s} \right) U_\tau + \overset{M}{\Delta}\, U_{\tau-1} \right\} + \log \Gamma \sum_{\tau=1}^{\mu} \Gamma^{\tau-1} \left\{ \right.$$
$$\left. \left\{ 4\, \tau \left(\tau + s \frac{\partial}{\partial\, s} \right) V_\tau + \overset{M}{\Delta}\, V_{\tau-1} \right\} + \sum_{\tau=1}^{m} \Gamma^{\tau-1} \left\{ 4 \left(m + 2\tau + s \frac{\partial}{\partial\, s} \right) V_\tau \right\} \right. .$$

For the case when the number of dimension n is odd, we obtain the lemma if we put here that $V_\tau = 0$, and U_τ determine successively so that

$$4\,(\tau - m) \left(\tau + s \frac{\partial}{\partial\, s} \right) U_\tau + \overset{M}{\Delta}\, U_{\tau-1} = 0 ,$$

where U_τ are infinitely differentiable functions of x in the vicinity of ξ and $U_0(\xi,\xi)=1$, $U_{-1}\equiv 0$. In this case we have

$$U_\tau(x,\xi) = -\frac{1}{4(\tau-m)}\frac{1}{s^\tau}\int_0 r^{\tau-1}\overset{M}{\Delta} U_{\tau-1}\,dr. \tag{2.2.16}$$

This formula may be expressed also in the system of normal coordinates where it will have a more simpler form

$$U_\tau(x,\xi) = -\frac{1}{4(\tau-m)}\int_0^1 t^{\tau-1}\overset{M}{\Delta} U_{\tau-1}(x(\xi,at),\xi)\,dt, \tag{2.2.17}$$

from where it is easy to see that $U_\tau(x,\xi)$ is of the class C^∞ with respect to x and ξ on $\mathfrak{G}$.

Now, we shall consider the case when the number of dimension n is even, and first we assume $n>2$. In this case we suppose the functions $U(x,\xi)$ and $V(x,\xi)$ in the form of following power series

$$\begin{aligned} U &= M\left(\sum_{\tau=0}^{m-1} U_\tau \Gamma^\tau + \sum_{\tau=0}^{\nu-m} U_{m+\tau}\Gamma^{m+\tau}\right), \\ V &= M\sum_{\tau=0}^{\mu} V_\tau \Gamma^\tau, \end{aligned} \tag{2.2.18}$$

with $U_0=1$ and $V_{-1}\equiv 0$. For sake of simplicity let $\mu=\nu-m$.

If now we apply the operator (2.2.12) to (2.2.1) in such a way as (2.2.14) and take into account (2.2.8) we obtain

$$\begin{aligned} M^{-1}\Delta\overset{*}{P} &= \sum_{\tau=1}^{m-1}\Gamma^{-m+\tau-1}\left\{4(\tau-m)\left(\tau+s\frac{\partial}{\partial s}\right)U_\tau+\overset{M}{\Delta}U_{\tau-1}\right\}+\log\Gamma\sum_{\tau=1}^{\nu}\Gamma^{\tau-1}\left\{4\tau\left(\tau+m+\right.\right. \\ &\left.\left. +s\frac{\partial}{\partial s}\right)V_\tau+\overset{M}{\Delta}V_{\tau-1}\right\} -\Gamma^{-1}\left\{4\left(m+s\frac{\partial}{\partial s}\right)V_0+\overset{M}{\Delta}U_{m-1}\right\}+\sum_{\tau=1}^{\nu-m}\Gamma^{\tau-1}\left\{4\tau\left(\right.\right. \\ &\left.\left(m+\tau+s\frac{\partial}{\partial s}\right)U_{m+\tau}+4\left(m+2\tau+s\frac{\partial}{\partial s}\right)V_\tau+\overset{M}{\Delta}U_{m+\tau-1}\right\}. \end{aligned}$$

From here we obtain successively

$$\begin{aligned} U_\tau &= -\frac{1}{4(\tau-m)}\frac{1}{s^\tau}\int_0^s r^{\tau-m}\overset{M}{\Delta}U_{\tau-1}\,dr, \qquad (\tau=1,\ldots,m-1) \\ U_m &= 0, \end{aligned}$$

$$V_0 = -\frac{1}{4\tau}\frac{1}{s^m}\int_0^s r^{m-1}\overset{M}{\Delta}U_{m-1}\,dr, \tag{2.2.19}$$

$$V_\tau = -\frac{1}{4\tau}\frac{1}{s^{\tau+m}}\int_0^r r^{\tau+m-1}\overset{M}{\Delta} V_{\tau-1}\,dr. \qquad (\tau = 1,3,\ldots,\mu),$$

$$U_{m+\tau} = -\frac{1}{4\tau}\frac{1}{s^{m+\tau}}\int_0^s r^{m+\tau-1}\left\{\overset{M}{\Delta} U_{m+\tau-1} - \frac{1}{\tau}\overset{M}{\Delta} V_{\tau-1} + 4\tau V_\tau\right\} dr$$

$$(\tau = 1,2,\ldots,\nu - m).$$

Finally, applying the operator (2.2.12) to (2.2.1) for the case $n = 2$ and taking (2.2.15) we obtain

$$M^{-1}\Delta \overset{*}{P} = \log\Gamma\sum_{\tau=1}^{\nu}\Gamma^{\tau-1}\left\{4\tau\left(\tau + s\frac{\partial}{\partial s}\right)U_\tau + \overset{M}{\Delta} U_{\tau-1}\right\} + \sum_{\tau=1}^{\nu}\Gamma^{\tau-1}\left\{4\tau\left(\tau + s\frac{\partial}{\partial s}\right)V_\tau + \right.$$

$$\left. + 4\left(2\tau + s\frac{\partial}{\partial s}\right)U_\tau + \overset{M}{\Delta} V_{\tau-1}\right\},$$

and from here successively

$$U_\tau = -\frac{1}{4\tau}\frac{1}{s^\tau}\int_0^s r^{\tau-1}\overset{M}{\Delta} U_{\tau-1}\,dr, \qquad (\tau = 1,2,\ldots,\nu),$$

(2.2.20) $V_0 = 0$,

$$V_\tau = -\frac{1}{4\tau}\frac{1}{s^\tau}\int_0^s r^{\tau-1}\left\{\overset{M}{\Delta} V_{\tau-1} - \frac{1}{\tau}\overset{M}{\Delta} U_{\tau-1} + 4\tau U_\tau\right\} dr. \qquad (\tau = 1,2,\ldots,\nu).$$

We can express all above cited formulae in the normal coordinates and in that case they will be more simplified. Also it is easily to be concluded that all of them belong to C^∞ with respect to x and ξ on $\mathfrak{G}$,

Now we shall prove the convergence of the series (2.2.15) and (2.2.18) by using the method of dominant functions.* To this purpose, we introduce the normal coordinates instead of x, observing that U_τ and V_τ are functions of the class C^∞ of ξ^i and $y^i = y^i(x, \xi)$. In the new coordinates the operator $\overset{M}{\Delta}$ will have the form

$$\overset{M}{\Delta} = \overset{M}{G}{}^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\,\partial y^\beta} + \overset{M}{A}{}^\alpha\frac{\partial}{\partial y^\alpha} + \overset{M}{B},$$

where $\overset{M}{G}{}^{\alpha\beta}$, $\overset{M}{A}{}^\alpha$ and $\overset{M}{B}$ are infinitely differentiable matrix functions of ξ and y, because g^{ij}, A^i and B belong to $C^\infty(\mathfrak{G})$, and these are obtained from them by transformation of coordinates by rules given at the beginning of the section 2.1. Every coefficient of the above operator allows in a certain small open set $\{\xi : r(\xi, \xi_0) < \varepsilon\} \subset \mathfrak{G}$ around the fixed point $\xi_0 \in \mathfrak{G}$ the dominant

$$\alpha\left(1 - \frac{\sigma}{\rho}\right)^{-1}, \qquad \sigma = \sum_{i=1}^{n} |y^i|,$$

* See Hadamard [5].

for suitably chosen positive constants α and ρ. For the dominant of the certain matrix function $D(\xi, y)$ represented in the form of the series

$$D(\xi,y) = D_0(\xi) + \frac{1}{1!} D_\alpha(\xi)\, y^\alpha + \frac{1}{2!} D_{\alpha\beta}(\xi)\, y^\alpha y^\beta + \cdots, \tag{2.2.21}$$

we understand the new function $d(y)$ such that the matrix functions $D_{\alpha\beta\ldots\delta}(\xi)$ of the class $C^\infty(\mathfrak{G})$ satisfy the inequality

$$|D_{\alpha\beta\ldots\gamma}(\xi)| \leqslant d_{\alpha\beta\ldots\gamma},$$

where $|\quad|$ denotes the norm, everywhere in the small open set around which is under consideration. The norm of the matrix D is defined by

$$|D| = (\sum |D_{\alpha\beta\ldots\delta}|^2)^{1/2}.$$

By $D \ll d$ we shall denote that the matrix D allows the dominant d. Now, we can verify that if

$$D(\xi, y) \ll k\left(1 - \frac{\sigma}{\rho}\right)^{-l}, \qquad (l \geqslant 1),$$

respectively

$$\overset{M}{\Delta} D \ll l(l+1)\,\alpha' k\left(1 - \frac{\sigma}{\rho}\right)^{-l-3}, \text{ where } \alpha' = \alpha\left(1 + \frac{n}{\rho} + \frac{n^2}{\rho^2}\right),$$

then for $\tau \leqslant l + 2$, we have

$$\frac{1}{s^\tau}\int_0^s r^{\tau-1} \overset{M}{\Delta} D \,\mathrm{d}r \ll \frac{l(l+1)\,\alpha' k}{\tau\left(1 - \frac{\sigma}{\rho}\right)^{l+3}}. \tag{2.2.22}$$

Let us go over to our case, namely the functions U and V given by the series (2.2.15) and (2.2.18). We shall first observe the case when n is odd. Starting from

$$U_0 = 1 \ll k_0\left(1 - \frac{\sigma}{\rho}\right)^{-1},$$

from (2.2.16) and (2.2.17) we see that

$$U_\tau \ll k_\tau\left(1 - \frac{\sigma}{\rho}\right)^{-2\tau-1}, \quad \text{where } k_\tau = 2\,m\,\alpha' k_{\tau-1}, \text{ and } k_0 = 1,$$

If we choose suitably the positive constant g, then obviously $\Gamma = y_i\, y^i \ll g\left(\frac{\sigma}{\rho}\right)^2$.

Hence, we have

$$U_\tau\, \Gamma^\tau \ll \left(\frac{\beta\sigma}{\rho}\right)^{2\tau}\left(1 - \frac{\sigma}{\rho}\right)^{-2\tau-1},$$

respectively

$$\sum_{\tau=0}^{\nu}\left(\frac{\beta\sigma}{\rho}\right)^{2\tau}\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-1}\ll\left(1-\frac{1+\beta}{\rho}\sigma\right)^{-1}, \tag{2.2.23}$$

where $\beta=(2m\alpha' g)^{1/2}$. proving the absolute and uniform convergence of the series $\sum U_\tau\Gamma^\tau$ in the mentioned domain around the point ξ_0. The series is convergent for $\sigma<\frac{\rho}{2(1+\beta)}$.

We can apply the same procedure also to the case n even and >2. We omit details about it but immediately go over to dominants. Namely

$$\begin{aligned} &U_\tau\ll\delta^\tau\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-1}, \qquad (\tau=0,1,\dots,m-1),\\ &V_\tau\ll\delta^{\tau+m}\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-2m-1}, \qquad (\tau=0,1,\dots,\mu), \end{aligned} \tag{2 2.24}$$

where $\delta=\frac{n-1}{2}\alpha'$. By induction we obtain

$$U_\tau\ll\tau\,\delta^{\tau-1}(\delta+1)\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-1}, \qquad (\tau=m,m+1,\dots,\nu). \tag{2.2.25}$$

Now from (2.2.24) we get

$$V_\tau\Gamma^\tau\ll\delta^m\left(\frac{\beta\sigma}{\rho}\right)^{2\tau}\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-2m-1},$$

respectively

$$\sum_{\tau=0}^{\nu}\left(\frac{\beta\sigma}{\rho}\right)^{2\tau}\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-2m-1}\ll\left(1-\frac{\sigma}{\rho}\right)^{-2m}\left(1-\frac{\sigma}{1+\beta}\,b\right)^{-1},\ \beta^2=\delta g, \tag{2.2.26}$$

proving the absolute and uniform convergence of the series $\sum V_\tau\Gamma^\tau$ in the previously mentioned domain for $\sigma<\rho/2(1+\beta)$,

In the same way from (2.2.25) for the case $\tau\geqslant m$ we have

$$U_\tau\Gamma^\tau\ll\tau\left(\frac{\beta\sigma}{\rho}\right)^{2\tau}\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-1},$$

respectively

$$\sum_{\tau=m}^{\nu}\tau\left(\frac{\beta\sigma}{\rho}\right)^{2\tau}\left(1-\frac{\sigma}{\rho}\right)^{-2\tau-1}\ll\left(\frac{\beta\sigma}{\rho}\right)\left(1-\frac{\sigma}{\rho}\right)^{-1}\left(1-\frac{1+\beta}{\rho}\sigma\right)^{-2}, \tag{2.2.27}$$

where $\beta^2=(1+\delta)g$. The expression (2.2.27) shows the absolute and uniform convergence of series $\sum U_\tau\Gamma^\tau$ in the previosly mentioned domain for $\sigma<\rho/2\ (1+\beta)$ In that way we have proved in fullness the convergence of the series (2.2.18).

Since, the first of (2.2.20) is equal te the first of (2.2.19), and the same is also valid for the last of (2.2.20) and (2.2.19), it is obviously that also for $n=2$ the series (2.2.15) converge and therefore, we shall not dwell on the proof.

Since we have proved the absolute and uniform convergence of the series (2.2.15) and (2.2.18) for $x, \xi \in \mathfrak{G}$ in a certain neighbourhood of $\xi_0 \in \mathfrak{G}$, we can finally conclude that the lemma is proved, respectively that the operator Δ applied to a mixed type tensor has a parametrix of the following form

$$(2.2.28)\quad P_j^i(x,\xi)=\begin{cases} -\dfrac{1}{4\pi}\log\Gamma(x,\xi)\cdot U_j^i(x,\xi)+V_j^i(x,\xi), & \mathrm{n}=2,\\ \dfrac{1}{(n-2)\tau_n}\Gamma^{-m}(x,\xi)\cdot U_j^i(x,\xi)+\log\Gamma(x,\xi)\cdot V_j^i(x,\xi), & n=4,6,8,\ldots,\\ \dfrac{1}{(n-2)\tau_n}\Gamma^{-m}(x,\xi)\cdot U_j^i(x,\xi), & n=3,5,7,\ldots, \end{cases}$$

such that the conditions (2.2.2) and (2.2.3) are satisfied.

2.3. The fundamental solution

Let $\bar{\mathfrak{E}}$ be the closure of the open set $\mathfrak{E}$ contained in an open set $\mathfrak{G} \subset \mathfrak{R}$ such that the parametrix $\overset{*}{P}(x,\xi)$ is defined for arbitrary $x, \xi \in \mathfrak{G}$ and let $\mathfrak{H}$ be an arbitrary small open subset of $\mathfrak{E}$ such that the closure $\bar{\mathfrak{H}}$ is also contained in $\mathfrak{E}$. Under these conditions we assume the following

Hypothesis. There exist a small positive constant γ with the following properties: Let $s(x,\xi)$ be the distance between the points $x \in \bar{\mathfrak{E}}$ and $\xi \in \bar{\mathfrak{H}}$ and let $\delta(s)$ be an infinitely differentiable function and $\geqslant 0$ for $s \geqslant 0$ such that

$$(2.3.1)\qquad \delta(s)=\begin{cases}1, & 0\leqslant s\leqslant\gamma\\ 0, & s\geqslant 2\gamma\end{cases}$$

then

a) the tensor function

$$(2.3.2)\qquad P_j^i(x,\xi)=\begin{cases}\dfrac{1}{(n-2)\tau_n}\mathrm{T}(\Gamma)\cdot U_j^i(x,\xi)+\Pi(\Gamma)\cdot V_j^i(x,\xi), & n>2,\\ -\dfrac{1}{4\pi}\Pi(\Gamma)\cdot U_j^i(x,\xi)+V_j^i(x,\xi), & n=2,\end{cases}$$

where

$$(2.3.3)\qquad \begin{aligned}\mathrm{T}(\Gamma)&=\delta(s)\,\Gamma^{-m},\\ \Pi(\Gamma)&=\delta(s)\log\Gamma,\end{aligned}$$

is defined everywhere;

b) the function

$$K_j^i(x, \xi) = \Delta\ P_j^i(x, \xi) \tag{2.3.4}$$

is of the class C^∞ ($\mathfrak{G}$) and is bounded;

c) the integral

$$\int_{s(x,\xi)\leq 2\gamma} \mathrm{d}x \qquad \text{is bounded in } \xi; \tag{2.3.5}$$

d) the integral

$$\int_{\mathfrak{G}} |P(x, \xi)|\,\mathrm{d}\,\xi \qquad \text{is bounded in } x. \tag{2.3.6}$$

The above hypothesis will be surely satisfied when $\mathfrak{G}$ is compact, and in the general case it will impose conditions upon the coefficients g^{ij}, A^i and B.

Theorem. *Let the assumed hypothesis be satisfied. Then the tensor function*

$$F_j^i(x, \xi) = P_j^i(x, \xi) + \int_{\mathfrak{G}} Q_k^i(x, \eta)\, P_j^k(\eta, \xi)\,\mathrm{d}\,\eta, \tag{2.3.7}$$

where

$$Q_k^i(x, \eta) = \sum_{r=1}^{\infty} K_{(r)k}^{\ i}(x, \eta), \tag{2.3.8}$$

$$K_{(1)k}^{\ i}(x, \eta) = K_k^i(x, \eta), \quad K_{(r)k}^{\ i}(x, \eta) = \int_{\mathfrak{G}} K_s^i(x, \zeta)\, K_{(r-1)k}^{\ s}(\zeta, \eta)\,\mathrm{d}\,\zeta$$

satisfies the equation

$$\Delta\, F_j^i(x, \xi) = 0, \qquad x, \xi \in \mathfrak{G}, \tag{2.3.9}$$

and the conditions (*2.1.8*) *and* (*2.1.9*).

Proof. We shall first prove the convergence of the series (2.3.8). To this purpose we observe Banach's space $C(\mathfrak{G})$, all real-valued continuous C^∞ tensor functions $T^{ij}(x.\xi)$ defined on $\mathfrak{G}$, with the norm

$$\|T\| = \sup_{x,\xi\in\mathfrak{G}} |T(x, \xi)|,$$

where

$$|T(x, \xi)| = (|T_{ij}(x, \xi)\, T^{ij}(x, \xi)|)^{1/2}.$$

If we now, by the expressions (2.3.4) and (2.3.5), put

$$\sup_{x,\xi\in\mathfrak{G}} |K(x, \xi)| = S, \qquad \sup_{\xi} \int_{s(x,\xi)\leq 2\gamma} dx = L,$$

then we have the following estimations

(2.3.10)
$$\sup_{x,\xi \in \mathfrak{G}} | K_{(r)}(x,\xi) | \leqslant S^n L^{n-1}$$
$$\sup_{\xi \in \mathfrak{G}} \int_{\mathfrak{G}} | K_{(r)}(x,\xi) | \, \mathrm{d}x \leqslant S^n L^n .$$

This proves the convergence of (2.3.8). Thus, $Q^i_k(x,\eta)$ is a continuous C^∞ tensor function of x and ξ on a bounded set $\mathfrak{G}$.

To prove that $F^i_j(x,\xi)$ satisfies (2.3.9), we must first show the possibility of the differentiation under the sign of the integral. Let $\mathfrak{E}$ be a small open set $\subset \mathfrak{G}$ with the regular closed boundary $\mathfrak{J}$ such that the parametrix $P^i_j(x,\xi)$ is defined for arbitrary $x,\xi \in (\mathfrak{E} \cup \mathfrak{J})$. We suppose that $\xi \in \mathfrak{E}$ is fixed and is the centre of a small geodesic sphere $S(\delta)$ of the radius δ and of the surface $\mathfrak{P}(\delta)$. Let us consider on the domain

$$\mathfrak{E} = \mathfrak{E}_\delta + S(\delta)$$

the integral

$$\int_{\mathfrak{E}} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{E} = \int_{\mathfrak{E}_\delta} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{E}_\delta + \int_{S(\delta)} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}S(\delta),$$

and let us make the differentiation with respect to ξ. Then

$$\partial_n \int_{\mathfrak{E}} = \int_{\mathfrak{E}_\delta} Q^i_k(x,\eta) \, \partial_n P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{E}_\delta + \int_{S(\delta)} Q^i_k(x,\eta) \, \partial_n P^k_j(\eta,\xi) \mathrm{d}S(\delta).$$

Since P^k_j is the continuous tensor function on $S(\delta) \cup \mathfrak{P}(\delta)$ and it has the continuous first derivative, and $\int_S Q^i_k(x,\eta) \, \partial_n P^k_j(\eta,\xi) \, \mathrm{d}S$ is convergent, then is valid the relation

(2.3.11)
$$\int_{S(\delta)} Q^i_k(x,\eta) \, \partial_n P^k_j(\eta,\xi) \, \mathrm{d}S(\delta) = \int_{\mathfrak{P}(\delta)} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{P}_n .$$

If one examines the behaviour of the above integral when it may be seen that is valid the estimation

$$\lim_{\delta \to 0} \int_{\mathfrak{P}(\delta)} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{P}_n = 0\,(\delta^{1/2}).$$

Thus

(2.3.12)
$$\partial_s \int_{\mathfrak{E}} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{E} = \int_{\mathfrak{E}} Q^i_k(x,\eta) \, \partial_s P^k_j(\eta,\xi) \, d\,\mathfrak{E},$$

and it is easy verify the also validity for the covariant derivative, namely that

(2.3.13)
$$\nabla_s \int_{\mathfrak{E}} Q^i_k(x,\eta) P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{E} = \int_{\mathfrak{E}} Q^i_k(x,\eta) \, \nabla_s P^k_j(\eta,\xi) \, \mathrm{d}\,\mathfrak{E}.$$

A once more differentiation in the same way as the above, namely $g^{ij}\partial_i \partial_j$ shows that we need to examine

$$\int_{\mathfrak{P}(\delta)} Q^i_k(x,\eta)\,\partial_s P^k_j(\eta,\xi)\,\mathrm{d}\,\mathfrak{P}^s. \tag{2.3.14}$$

First we shall give certain estimations. Namely, since $\Gamma=\delta^2=\text{const}$ on $\mathfrak{P}(\delta)$, then the surface element $d\mathfrak{P}_i$ on $\mathfrak{P}(\delta)$ is parallel to the vector $\partial_i \Gamma=\Gamma_i$. Hence, by taking that

$$g^{ij}\Gamma_i\Gamma_j=4\,\Gamma$$

$$\mathrm{d}\,\mathfrak{P}=g_{ij}\,\mathrm{d}\,\mathfrak{P}^i\,\mathrm{d}\,\mathfrak{P}^j,$$

we obtain the following relations

$$\begin{aligned} \Gamma_i\,\mathrm{d}\,\mathfrak{P}_k&=\Gamma_k\,\mathrm{d}\mathfrak{P}_i,\\ \Gamma^i\,\mathrm{d}\,\mathfrak{P}_i&=\sqrt{4\,\Gamma}\,\mathrm{d}\,\mathfrak{P}\,. \end{aligned} \tag{2.3.15}$$

From (2.2.28) we see that the parametrix $P^i_j(x,\xi)$ has the following form

$$P^i_j(\eta,\xi)=k\,l(\Gamma)\,Y^i_j(\eta,\xi)\,,$$

where $l(\Gamma)$ is either Γ^{-m} or $\log\Gamma$, and $Y^i_j(\eta,\xi)$ is either $U^i_j(\eta,\xi)$ or $V^i_j(\eta,\xi)$. Substituting it into (2.3.14) we obtain

$$k\int_{\mathfrak{P}(\delta)} Q^i_k(x,\eta)\,l'\,Y^k_j(\eta,\xi)\,\Gamma^s\,\mathrm{d}\,\mathfrak{P}_s+k\int_{\mathfrak{P}(\delta)} Q^i_k(x,\eta)\,\partial_s\,Y^k_j(\eta,\xi)\,l\,\mathrm{d}\,\mathfrak{P}^s\,. \tag{2.3.16}$$

When $l(\Gamma)=\Gamma^{-m}$ then the first integral in (2.3.16), by taking (2.3.15), gives

$$\lim_{\delta\to 0}\int_{\mathfrak{P}(\delta)} k\,Q^i_k(x,\eta)\,U^k_j(\eta,\xi)\,l'\,\Gamma_s\,\mathrm{d}\,\mathfrak{P}^s=-(n-2)\,\tau_n\,k\,Q^i_k(x,\eta)\,U^k_j(\xi,\xi),$$

and the second one

$$\lim_{\delta\to 0}\int_{\mathfrak{P}(\delta)} k\,Q^i_k(x,\eta)\,\partial_s\,Y^k_j(\eta,\xi)\,l\,\mathrm{d}\,\mathfrak{P}^s=0\,(\delta^{1/2})\,.$$

Also in the other case when $l(\Gamma)=\log\Gamma$ we can readily verify that the estimations for (2.3.14)

$$\lim_{\delta\to 0}\int_{\mathfrak{P}(\delta)}=\begin{cases}0, & n>2,\\ 4\pi k\,Q^i_k(x,\xi)\,U^k_j(\xi,\xi), & n=2,\end{cases}$$

are valid.

On the base of all said and taking that $U^i_j(\xi,\xi)=\delta^i_j$, and that

$$k=\begin{cases}\dfrac{1}{(n-2)\,\tau_n}, & n>2,\\[2mm] -4\pi, & n=2.\end{cases}$$

we get finally the estimation for (2.3.14)

$$\lim_{\delta\to 0} \int_{\mathfrak{P}(\delta)} Q_k^i(x,\eta)\,\partial_s P_j^k(\eta,\xi)\, d\mathfrak{P}^s = \begin{cases} -Q_j^i(x,\xi), & n>2, \\ -Q_j^i(x,\xi), & n=2. \end{cases}$$

Thus

$$g^{ns}\partial_n\partial_s \int_{\mathfrak{E}} Q_k^i(x,\eta)\,P_j^k(\eta,\xi)\,d\mathfrak{E} = \int_{\mathfrak{E}} Q_k^i(x,\eta)\,g^{ns}\partial_n\partial_s P_j^k(\eta,\xi)\,d\mathfrak{E} - Q_j^i(x,\xi), \tag{2.3.17}$$

respectively in the case of covariant derivatives, since $g^{ns}\nabla_n\nabla_s = \Delta$, we have

$$\Delta \int_{\mathfrak{E}} Q_k^i(x,\eta)\,P_j^k(\eta,\xi)\,d\mathfrak{E} = -Q_j^i(x,\xi) + \int_{\mathfrak{E}} Q_k^i(x,\eta)\,\Delta P_j^k(\eta,\xi)\,d\mathfrak{E}. \tag{2.3.18}$$

We can now easily prove (2.3.9). Applying the operator to (2.3.7)

$$\Delta F_j^i = \Delta P_j^i + \Delta(Q_k^i \circ P_j^k),$$

where is introduced the symbolic product

$$(Q_k^i \circ P_j^k)(x,\eta) = \int_{\mathfrak{G}} Q_k^i(x,\eta)\,P_j^k(\eta,\xi)\,d\mathfrak{G},$$

because of (2.3.4), (2.3.8) and (2.3.18) we have

$$\Delta F_j^i = K_j^i - Q_j^i + Q_k^i \circ K_j^k = 0\,.$$

We have also (2.1.8) – (2.1.9) by applying Fubini's theorem. Thus the proof of the theorem is completed.

3. The linearized Navier-Stokes equations

In this Chapter we shall consider the slow streaming, respectively the current of strong viscous fluids, therefore, we can in the equation (1.1) neglect the influence of inertial terms. In this case, hence, on $\mathfrak{D}$ respectively on $\mathfrak{D}_R$ exist the following equations

$$\rho F^i + t^{ij}_{,j} = 0, \qquad v^i_{,i} = 0, \tag{3.1}$$

respectively

$$\rho F^i - p_{,j}\,g^{ij} + \mu\,\Delta v^i = 0, \tag{3.2}$$

$$v^i_{,i} = 0, \tag{3.3}$$

with the conditions (1.3)–(1.5).

From now on when we do not speak strictly that anything is related to the domain $\mathfrak{D}$ or $\mathfrak{D}_R$ then we shall always turn to any open set $\mathfrak{G} \subset \mathfrak{R}$. The boundary of the set $\mathfrak{G}$ we denote with $\mathfrak{F}$, and the boundary of the domain $\mathfrak{D}$ with $\mathfrak{B}$. Accordingly to the Riemannian metric we naturally denote the measure on a set $\mathfrak{G}$ with $d\mathfrak{G}$, where

$$d\mathfrak{G} = (g(x))^{1/2}\,dx^1\cdots dx^n, \qquad \text{and} \qquad g(x) = \det(g_{ij}(x)).$$

Moreover, we shall assume that the space $\mathfrak{G}$ is flat and entirely filled with fluid. Under mentioned suppositions let us solve the Dirichlet's problem.

3.1. *The method of potentials of distributions*

If we neglect in the equation (3.2) the influence of the extraneous force field F^i, then we can state the following

Theorem. *The Dirichlet's problem for (3.2) and (3.3) is uniquely solvable in the class $C^\infty(\mathfrak{D})$ at the continous field $f^i \in C^\infty(\mathfrak{B})$, satisfying the condition (1.5). The problem is also uniquely solvable in the class $C^\infty(\mathfrak{D}_R)$ at the arbitrary continuous field $f^i \in C^\infty(\mathfrak{B})$, and for $n \geqslant 2$ odd. The solutions of the class $C^\infty(\mathfrak{G})$ completely continuous up to the boundary $\mathfrak{B}$ are given in the form of potentials of double distributions*

$$v^i(x) = \int_{\mathfrak{B}} \nabla_k v^{ij}(x, \xi)\, \rho^k(\xi)\, d\mathfrak{B}_j, \tag{3.1.1}$$

$$p(x) = \int_{\mathfrak{B}} \nabla_k p^j(x, \xi)\, \rho^k(\xi)\, d\mathfrak{B}_j, \tag{3.1.2}$$

where $x \in \mathfrak{G}$, $\xi \in \mathfrak{B}$, *and*

$$\rho^i(\eta) = \overset{*}{f}{}^i(\eta) + \sum_{\nu=1}^{\infty} \lambda^\nu K^{\,i}_{(\nu)}(\eta), \tag{3.1.3}$$

$$K^{\,i}_{(\nu)}(\eta) = \int_{\mathfrak{B}} K^{\,i}_{(\nu)\,j}(\eta, \zeta)\, \overset{*}{f}{}^j(\zeta)\, d\mathfrak{B},$$

$$K^{\,i}_{(1)\,k} = K^{\,i}_{k} = \nabla_k\, v^{ij} n_j, \qquad K^{\,i}_{(\nu)\,j}(\eta, \zeta) = \int_{\mathfrak{B}} K^{\,i}_{(1)\,k}(\eta, \xi)\, K^{\,k}_{(\nu-1)\,j}(\xi, \zeta)\, d\mathfrak{B},$$

at which $\lambda = \pm 2$; $\overset{*}{f} = \pm 2f$ *corresponds to the case in* $\mathfrak{D}$ *and* $\mathfrak{D}_R$ *respectively*

Proof. Any tensor function

$$v^{ik} = g^{ik} S + g^{jk} \nabla_j \varphi^i, \tag{3.1.4}$$

from the class $C^\infty(\mathfrak{G})$, will satisfy the equations (3.2) and (3.3) if

$$\Delta S = 0, \tag{3.1.5}$$

$$\Delta \varphi^i = -g^{ki} \nabla_k S, \tag{3.1.6}$$

$$p^i = \mu\, \Delta \varphi^i. \tag{3.1.7}$$

This can be very easily verified. Thus $v^{ik}(x, \xi) \in C^\infty(\mathfrak{G})$ represents a fundamental solution-fundamental tensor of the equations (3.2) and (3.3). In the purpose of a fundamental solution we can also use a parametrix. The fundamental solution of the equation (3.1.5) we had already given in Chapter 2 by (2.3.7), that only instead of the tensor field, in this case a scalar field stands. But, the solution of the equation (3.1.6) we shall assume in the form

$$\varphi^i(\eta, \zeta) = \int_{\mathfrak{G}} F^{ij}(\eta, \xi)\, \nabla_j S(\xi, \zeta)\, d\mathfrak{G}, \tag{3.1.8}$$

where $F^{ij}(\eta, \xi)$ is given by (2.3.7). From (2.3.18) it can be readily verified that (3.1.8) satisfies (3.1.6). Afterwards, we have shown in Chapter 2 that $S \in C^\infty(\mathfrak{G})$, and from (3.1.8) follows also $\varphi^i \in C^\infty(\mathfrak{G})$ and so $v^{ij} \in C^\infty(\mathfrak{G})$.

To show that (3.1.1) and (3.1.2) are indeed the solutions of the stated problem we need before all to examine the behaviour of the potential when the point $x \in \mathfrak{G}$ approaches the point $\xi \in \mathfrak{B}$. To this purpose, we describe a small geodesic sphere $S(\delta)$ with the radius δ and the surface $\mathfrak{A}(\delta)$ about the point $\xi \in \mathfrak{B}$, and then we consider on the boundary

$$\mathfrak{B}_\delta = \mathfrak{B} \pm \frac{1}{2} \mathfrak{A}(\delta),$$

the expression

$$\int_{\mathfrak{B}_\delta} \nabla_k v^{ij}(x, \xi)\, \rho^k(\xi)\, d\mathfrak{B}_{\delta j} \mp \frac{1}{2} \int_{\mathfrak{A}(\delta)} \nabla_k v^{ij}(x, \xi)\, \rho^k(\xi)\, d\mathfrak{A}_j .$$

As in the section 2.3 we can also consider the limit value of the above integrals when $\delta \to 0$. As a result of the limit process the second integral on the right hand gives $-\rho^i(\xi)$. Thus, it can be concluded that

$$\overset{+}{v}{}^i(\xi) = \frac{1}{2} \rho^i(\xi) + v^i(\xi),$$

$$\overset{-}{v}{}^i(\xi) = -\frac{1}{2} \rho^i(\xi) + v^i(\xi),$$

and since on $\mathfrak{B}$ is given a field $f^i(\xi)$, obviously that the field $\rho^i(\xi)$ must satisfy the following integral equation

$$\rho^i(\xi) + \lambda \int_{\mathfrak{B}} \nabla_k v^{ij}(\xi, \eta)\, \rho^k(\eta)\, d\mathfrak{B}_j = \overset{*}{f}{}^i(\xi), \tag{3.1.9}$$

in which $\lambda = 2$ and $\overset{*}{f} = 2f$ denote that it is an interior problem, respectively in $\mathfrak{D}$, and $\lambda = -2$ and $\overset{*}{f} = -2f$ denote that it is an exterior one respectively in $\mathfrak{D}_R$.

If we introduce the symbol

$$K^{ij}(\eta, \xi) = \nabla_k v^{ij}(\eta, \xi)\, n^k, \tag{3.1.10}$$

where n^k is the normal vector to $\mathfrak{B}$, then the equation (3.1.9) obtains the new form

$$\rho^i(\xi) + \lambda \int_{\mathfrak{B}} K^{ij}(\xi. \eta)\, \rho_j(\eta)\, d\mathfrak{B} = \overset{*}{f}{}^i(\xi). \tag{3.1.11}$$

An iteration procedure of solving the above equation leads to the Neumann series

$$\rho^i(\xi) = \overset{*}{f}{}^i(\xi) + \sum_{\nu=1}^{\infty} \lambda^\nu K^i_{(\nu)}(\xi), \tag{3.1.12}$$

where

$$K^i_{(\nu)}(\xi) = \int_{\mathfrak{B}} K^{ij}_{(\nu)}(\xi, \eta)\, \overset{*}{f}_j(\eta)\, d\mathfrak{B}, \tag{3.1.13}$$

is an iterated kernel, which for $\nu > 1$ has the form

$$K_{(\nu)}{}^{i}{}_{j}(\xi, \eta) = \int_{\mathfrak{B}} K_{(1)}{}^{i}{}_{k}(\xi, \zeta)\, K_{(\nu-1)}{}^{k}{}_{j}(\zeta, \eta)\, d\mathfrak{B}, \tag{3.1.14}$$

where $K_{(1)}{}^{i}{}_{k} = K^{i}_{k}$.

Substituting (3.1.13) into (3.1.12) and supposing for this time that we can reverse the order of summation and integration, then we obtain

$$\rho^{i}(\xi) = \overset{*}{f}{}^{i}(\xi) + \int_{\mathfrak{B}} h^{i}_{j}(\xi, \eta)\, \overset{*}{f}{}^{j}(\eta)\, d\mathfrak{B}, \tag{3.1.15}$$

where

$$h^{i}_{j}(\xi, \eta) = \sum_{\nu=1}^{\infty} \lambda^{\nu} K_{(\nu)}{}^{i}{}_{j}(\xi, \eta), \tag{3.1.16}$$

is a resolvent.

Let us prove the convergence of (3.1.12). To this purpose we consider the class $\mathfrak{L}^2$ of all measurable fields φ, such that the components $\varphi^{ij}(t)$ are defined for every t and are measurable functions of the coordinates $x^i = x^i(t)$, and $|\varphi|^2$ is integrable in the Lebesgue sense on $\mathfrak{G}$. Let L^2 be the linear space consisting of all measurable fields on the open set $\mathfrak{G}$ such that

$$(\varphi, \varphi) < +\infty,$$

i.e

$$L^2 = \{\varphi : (\varphi, \varphi) < +\infty\}.$$

L^2 constitutes a real Hilbert space on $\mathfrak{G}$, having (φ, ψ) as inner product. The norm in $L^2(\mathfrak{G})$ will be

$$\|\varphi\|_2 = \|\varphi\|_{2,\mathfrak{G}} = \{\int_{\mathfrak{G}} |\varphi(x)|^2 d\mathfrak{G}\}^{1/2}, \tag{3.1.17}$$

where $|\ \ |$ is the absolute value defined as follows

$$|\varphi(x)| = (|\varphi_{ij}(x)\, \varphi^{ij}(x)|)^{1/2}. \tag{3.1.18}$$

Denote by q the real number associated with $p = 2$ by the relation $\frac{1}{p} + \frac{1}{q} = 1$. Then, if $\|\varphi\|_p$ and $\|\psi\|_q$ are finite, the integral

$$(\varphi, \psi)_{\mathfrak{G}} = \int_{\mathfrak{G}} \varphi_{ij} \psi^{ij} d\mathfrak{G}, \tag{3.1.19}$$

converges absolutely and satisfies the Schwarz's inequality

$$|(\varphi, \psi)_{\mathfrak{G}}| \leqslant \|\varphi\|_p \cdot \|\psi\|_q. \tag{3.1.20}$$

Let us now consider the kernel $K^{ij}(\eta, \xi) \in \mathfrak{L}^2(\mathfrak{B})$. Such a kernel defines a bounded operator K on $L^2(\mathfrak{B})$ in the following way: If $\overset{*}{f} \in L^2(\mathfrak{B})$ let

$$F^{i}(\eta) = \int_{\mathfrak{B}} K^{ij}(\eta, \xi)\, \overset{*}{f}_{j}(\xi)\, d\mathfrak{B}. \tag{3.1.21}$$

Since

$$|F(\eta)|^2 \leqslant \int_{\mathfrak{B}} |K(\eta,\xi)|^2 d\xi \int_{\mathfrak{B}} |\overset{*}{f}(\xi)|\, d\xi, \qquad \xi \in \mathfrak{B}.$$

then integrating it with respect to $\eta \in \mathfrak{B}$, we see that

$$\|F\| = \|K \circ \overset{*}{f}\| \leqslant \{\int_{\mathfrak{B}}\int_{\mathfrak{B}} |K(\eta,\xi)|^2 d\eta\, d\xi\}^{1/2} \|\overset{*}{f}\|, \tag{3.1.22}$$

respectively

$$\|K\| \leqslant \{\int_{\mathfrak{B}}\int_{\mathfrak{B}} |K(\eta,\xi)|^2 d\eta\, d\xi\}^{1/2}. \tag{3.1.23}$$

By Fubini's theorem

$$\int_{\mathfrak{B}} |K(\eta,\xi)|^2 d\xi,$$

exists almost everywhere and is a square integrable function of $\eta \in \mathfrak{B}$. Proceeding inductively (3.1.22) we find that

$$(K_{(\nu)} \circ \overset{*}{f})^2 \leqslant \|\overset{*}{f}\|^2 H^{2(n-1)} \int_{\mathfrak{B}} |K(\eta,\xi)|^2 d\xi,$$

and from here

$$\|K_{(\nu)} \circ \overset{*}{f}\| \leqslant \|\overset{*}{f}\| H^n,$$

where

$$H = \{\int_{\mathfrak{B}}\int_{\mathfrak{B}} |K(\eta,\xi)|^2 d\eta\, d\xi\}^{1/2}.$$

Thus, for the series (3.1.12) is valid the inequality

$$\|\rho\| \leqslant \|\overset{*}{f}\| \left(1 + \sum_{\nu=1}^{\infty} |\lambda|^\nu H^\nu\right),$$

which shows that if f is square integrable and

$$|\lambda| H < 1, \tag{3.1.24}$$

that the Neumann series (3.1.22) converges in the mean to a square integrable vector function ρ^i which satisfies (3.1.11) almost everywhere on $\mathfrak{B}$. The above inequality also enables us to conclude that (3.1.16) is a $\mathfrak{L}^2$ kernel, that (3.1.12) converges almost everywhere, and that we could reverse the order of summation and of integration to obtain (3.1.15). Thus, obviously ρ^i is unique, that there exist the resolvent h^{ij} given by (3.1.16), as a bounded linear operator and that the solution of the integral equation (3.1.11) can be found in the form (3.1.15), i.e.

$$\rho^i(\xi) = ((I + H)\overset{*}{f}{}^i)(\xi),$$

where I is the indentity operator.

We shall now prove the complete continuity of the operator K, having the norm given by (3.1.23). From (2.1.7) we can formally conclude that, if $\Gamma(\eta,\xi)\to 0$, then the kernel (3.1.10) shows the singularity given by the representation

$$\frac{1}{\Gamma^{m+1}}\Gamma_k n^k \mathring{U}^{ij}(\eta,\xi)+\frac{\log\Gamma}{\Gamma}\Gamma_k n^k \mathring{V}^{ij}(\eta,\xi), \tag{3.1.25}$$

where $\mathring{U}^{ij}(\eta,\xi)$ and $\mathring{V}^{ij}(\eta,\xi)$ are continuous C^∞ tensor functions. Suppose the existence of a small positive constant $\alpha<\gamma$ with the following properties

$$K_1^{ij}(\eta,\xi)=\begin{cases}K^{ij}(\eta,\xi), & s\geqslant\alpha,\\ 0, & s<\alpha,\end{cases}\qquad K_2^{ij}(\eta,\xi)=\begin{cases}0, & s\geqslant\alpha,\\ K^{ij}(\eta,\xi), & s<\alpha.\end{cases} \tag{3.1.26}$$

Thus, we obtain the decomposition of the kernel.

Since on the bounded set $\mathfrak{G}$ the kernel $K_1^{ij}(\eta,\xi)$, where $\eta,\xi\in\mathfrak{B}$, is bounded, it only remains to show the estimation for $K_2^{ij}(\eta,\xi)$. From (3.1.21) introducing the representative

$$\frac{k^{ij}(\eta,\xi)}{\Gamma^{\lambda/2}}=\frac{k^{ij}(\eta,\xi)}{s^\lambda},\qquad 0<\lambda<n, \tag{3.1.27}$$

for (3.1.23) we obtain then

$$K_2^{ij}\circ f=\int_{\mathfrak{B}_\alpha} k^{ij}(\eta,\xi)\frac{f_j(\xi)}{s^\lambda}\,d\xi,\qquad \eta\in\mathfrak{B},$$

respectively

$$|K_2\circ f|^2\leqslant L^2\left[\int_{\mathfrak{B}_\alpha}\frac{|f|}{s^\lambda}\,d\xi\right]^2,$$

where $L=\sup_{\eta\in\mathfrak{B}}\int_{\mathfrak{B}_\alpha}|k(\eta,\xi)|\,d\xi$. Schwarz's inequality and Fubini's theorem finally give the estimation for the operator K_2, namely

$$\|K_2\|^2\leqslant L^2\left[\int_{\mathfrak{B}}\frac{1}{s^\lambda}d\eta\right]\left[\int_{\mathfrak{B}_\alpha}\frac{1}{s^\lambda}d\xi\right]. \tag{3.1.28}$$

The first integral on the right is bounded, and the second may be made as small as desired, namely it may be made arbitrary small for sufficiently small α. In this way the continuity of the operator K on $L^2(\mathfrak{B})$ is proved. We can now prove that the operator K is compact. Namely, if $\alpha=\frac{1}{N}$ and instead of $K_1\in\mathfrak{L}^2(\mathfrak{G})$ we define a new operator K_N, then it can be easily verified that

$$\|K-K_N\|\to O\quad\text{when}\quad N\to\infty.$$

The fact that $K_N\to K$ then assures us that K is compact.

Now we can give the estimations for the solutions (3.1.1) and (3.1.2). Namely, if the region $\mathfrak{G}$ is bounded then for the cited solutions are valid the following estimations

$$|v(x)| \leqslant C \| \rho \|_{L^2},$$

(3.1.29)

$$|p(x)| \leqslant C_1 \| \rho \|_{L^2},$$

where

$$C = \sup_{x \in \mathfrak{B}} \int_{\mathfrak{B}} |K(x, \xi)|^2 d\xi,$$

$$C_1 = \sup_{x \in \mathfrak{G}} \int_{\mathfrak{B}} |\bar{K}(x, \xi)|^2 d\xi.$$

These estimations show the continuity of the solutions in the whole region $\mathfrak{G}$. That these solutions belong to the class $C^\infty(\mathfrak{G})$ it is obviously as $K^{ij} \in (\mathfrak{G} \cup \mathfrak{F})$.

Since, we have shown the continuity of the solutions in the bounded region we shall now prove also its uniqueness on $\mathfrak{G}$. To this purpose we consider the homogenous equation of (3.1.9), i. e

$$\rho^i(\xi) + 2\int_{\mathfrak{B}} K^{ij}(\xi, \eta)\, \rho_j(\eta)\, d\eta = 0, \tag{3.1.30}$$

because $\lambda = 2$ corresponds to an interior problem, respectively in $\mathfrak{G}$. It can be easily verified that the above equation has n non-trivial linear independent solutions. Namely, the normal vector on n^i satisfies (3.1.30). It follows

$$n^i(\xi) + 2\, n^k(\xi) \int_{\mathfrak{B}} \nabla_k v^{ij}(\xi, \eta)\, d\mathfrak{B}_j,$$

and since

$$\int_{\mathfrak{B}} \nabla_k v^{ij}(\xi, \eta)\, d\mathfrak{B}_j = -\frac{1}{2} \delta^i_k,$$

which can be easily concluded from the behaviour of the above integral at $\eta \to \xi$, then

$$n^i(\xi) + 2 n^k(\xi) \left(-\frac{1}{2} \delta^i_k \right) = 0$$

This verifies our statement. That this is the only solution it can be easily shown, because every other solution will be linearly expressed by n^i. On the base of all said we conclude: In order that the inhomogenous equation (3.1.9) possesses a solution, it is necessary and sufficient that $\overset{*}{f^i}$ be orthogonal to the eigenfunctions (non-trivial solutions) n^i of the homogenous equation, i.e

$$\int_{\mathfrak{B}} \overset{*}{f^i} n_i\, d\mathfrak{B} = 0. \tag{3.1.31}$$

In this way we have proved the first part of the theorem, namely the case of a bounded region $\mathfrak{G}$ (i.e $\mathfrak{D}$).

Let us consider now the case of an infinite region $\mathfrak{G}$. The kernel $K^{ij}(x, \xi) \in C^{\infty}(\mathfrak{G} \cup \mathfrak{F})$ has a local character, namely it is valid for two neighbour points in a small open set $\mathfrak{G}_k$. But, as it is valid for every open subset $\mathfrak{G}_1$. $\mathfrak{G}_2, \ldots,$ belonging to a given system of neighbourhoods $\{\mathfrak{G}_k\}$, then it has a global character, i. e., it will be valid for the whole space $\mathfrak{G}$, covered by $\{\mathfrak{G}_k\}$. Suppose now the existence of a sequence of succesively continuous measurable fields $\{T_s\}$ on open sets $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \ldots \subset \mathfrak{E}_s = \mathfrak{G}$. The topology for each space $\mathfrak{E}_s$ of the sequence $\mathfrak{E}_1, \ldots, \mathfrak{E}_s$ is determined by a system neighbourhoods $\{\mathfrak{E}_{1j}\}, \ldots, \{\mathfrak{E}_{sl}\}$ such that $\mathfrak{E}_{1j} \subset \ldots \subset \mathfrak{E}_{sl}$. Namely, the topology in $\mathfrak{E}_s$ is induced by the metric of that space, and $\{\mathfrak{E}_{sl}\}$ forms a basis of the topology of $\mathfrak{E}_s$. At mapping of a field T_n on a set $\mathfrak{E}_n$ to a field T_m on a set $\mathfrak{E}_m$ we shall have also the change of the structure of the space $\mathfrak{E}_n$ to the space $\mathfrak{E}_m$, namely $g_{ij}^{(n)} \to g_{ij}^{(m)}$. The structure is continuously changeable to a sphere S of the infinitely large radius R. Locally looking, in this continuous process a geodesic sphere is transformed to Euclidean sphere. In that case the square of the geodesic distance gets

$$\Gamma(x, \xi) = G_{ij}(x)(x^i - \xi^i)(x^j - \xi^j), \tag{3.1.32}$$

where G_{ij} corresponds to the structure of the sphere S.

Let us now examine if our operator $K \in L^2(\overline{\mathfrak{G}})$ does a continuous mapping of a field T_s through a sequence of succesive fields $\{T_s\}$. Then, obviously we need to examine $\lim\limits_{s \to \infty} KT_s$. Since the mapping of the field T_s from a set to a set accompanies a change of the structure of the sequence $\{\mathfrak{E}_s\}$ of spaces up to a sufficiently large sphere, then the above limit process corresponds to the process $\lim\limits_{R \to \infty} KT_R$, i. e., letting that the point ξ is removed infinitely from x. By taking (3.1.25) we can give the following estimation

$$|v(x)| \leqslant M_1 N_1 \int\limits_{\mathfrak{B}} \frac{1}{s^{n-1}} \, d\xi + M_2 N_2 \int\limits_{\mathfrak{B}} \frac{\log s}{s} \, d\xi, \tag{3.1.33}$$

where

$$M = \sup_{\substack{x \in \mathfrak{G} \\ \xi \in \mathfrak{B}}} |k(x. \xi)|, \qquad N = \sup_{\xi \in \mathfrak{B}} |\rho(\xi)|.$$

Thus, when the number of dimensions of the space $\mathfrak{G}$ is odd, then for $|v|$ we have the estimation

$$|v| \leqslant A \frac{1}{s^{n-2}}, \tag{3.1.34}$$

namely, the sequence $\{v_N\}$, where $v_N = A \dfrac{1}{N^{n-2}}$ is decreasing, and when $N \to \infty$ then $v_N \to 0$. What means that, for any $\varepsilon > 0$, there exists a compact set $\mathfrak{G} \subset \mathfrak{R}$ such that $x \notin \mathfrak{G}$ implies $|v(x)| < \varepsilon$.

But, if the number of dimensions is even, then is valid the estimation

$$|v| \leqslant A \frac{1}{s^{n-2}} + B \log s. \tag{3.1.55}$$

Hence, the sequence $\{v_N\}$ has a logarithmic growth as $N \to \infty$.

In the same way it can be given the estimation also for the pressure field, namely

$$|p| \leqslant C \frac{1}{s^{n-1}}, \tag{3.1.36}$$

from where it is seen that $\{p_N\}$ does a decreasing sequence, namely $p(x)$ vanishes outside a compact set of x for every $n \geqslant 2$.

We can now examine the uniqueness of solutions for the case of an unbounded region $\mathfrak{G}$. Observe the associated homogenous equation of the equation (3.1.9),

$$\rho^i(\xi) - 2 \int_{\mathfrak{B}} K^{ij}(\xi, \eta)\, \rho_j(\eta)\, d\mathfrak{B} = 0, \tag{3.1.37}$$

because $\lambda = -2$ corresponds to an exterior problem, respectively in $\mathfrak{D}_R$. In the vicinity of infinity, since $p = 0$, then the equation of the motion is reduced to the form

$$D^{ij}{}_{,j} = 0 \quad \text{respectively} \quad \Delta v^i = 0, \tag{3.1.38}$$

which corresponds to a perfect fluid motion. In this case the deformation tensor becomes equal to zero, i. e.,

$$D^{ij} = 0. \tag{3.1.39}$$

It is easily seen that every solution of the equation (3.1.39), having $n(n+1)/2$ linearly independent, satisfies the equation (3.1.37). Namely, the covariantly differentiating equation (3.1.37)

$$\nabla_s \rho^i - 2 \int_{\mathfrak{B}} \nabla_s \nabla_k v^{ij} \rho_j n^k \, d\mathfrak{B} = 0,$$

and writing a new equation in which the indices i and s are changed, then if we make the addition of them taking (3.1.39) we obtain

$$\rho^{i,s} + \rho^{s,i} = 0, \tag{3.1.40}$$

which means that (3.1.37) has $n(n+1)/2$ linearly indenpendent solutions, which we shall denote by

$$\overset{i}{\rho}_{(k)}. \qquad k = 1, \ldots, n(n+1)/2,$$

where k denotes the ordinal number of that solution. The condition of solvability of the inhomogenous equation (3.1.9) is reduced to the orthogonality condition of $\overset{*}{f}{}^i$ and $\overset{i}{\rho}_{(k)}$, i.e.

$$\int \overset{*}{f}_i \, \overset{i}{\rho}_{(k)} \, d\mathfrak{B} = 0, \qquad k = 1, \ldots, n(n+1)/2. \tag{3.1.41}$$

In the case when the number of dimensions of the space $\mathfrak{G}$ is odd, the above condition is satisfied, but if n is even, the condition cannot be satisfied. For example, if $n = 2$ then $\overset{*}{f}{}^i$ $(i = 1,2)$ cannot be orthogonal to $\overset{i}{\rho}_{(k)}$ $(k = 1,2,3)$, and this is seen also for every other even n, i.e., $n = 4, 6, \ldots$

For the completing of the proof of the theorem we have the

Remark. *When the space $\mathfrak{S}$ possesses an odd number of dimensions, then an exterior problem is uniquely solvable, but if n is even then the satisfaction of the condition* (3.1.41) *is impossible and therefore, it is indispensable that v^i has a logarithmic growth in the vicinity of infinity.*

At the end we shall remark that by functional method generalized solutions of the Nevier-Stokes equations may be defined, but it we will not speak about it here.

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