PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 6 (20), 1966. p. 125—130

TOPOLOGIES ON COLLECTIONS OF CLOSED SUBSETS[1)]

M. Marjanović

(Communicated Jan. 24, 1966)

1. Introduction. In this paper we consider the collection of non-empty closed subsets of a topological space X, often called hyperspace of X, topologized by means of collections of open subsets submitted to the axiom (α_1) below and, in dealing with some other properties of these topologies, also to the axioms (α_2) and (α_3). According to E. Michael [3], the hyperspace of X shall be denoted by 2^x and we shall follow Michael in terminology and notations as well as in the way of introducing topologies on 2^x, although our approach is axiomatical and modified so to include various topologies on 2^x and on its subsets. The main result of this paper is the Theorem stating when 2^x is compact provided X is compact and which has various consequences (among them the Tychonoff Theorem on compactness of the product space).

2. Way of topologizing 2^x and some examples. We start with some definitions. An arbitrary element from 2^x is denoted by F and a fixed one by F_0. U stands for an open subset of X and $\mathcal{U}=\{V\}$ for a collection of such subsets.

A collection of open sets in X is a cover of the set $F \in 2^x$ denoted by $\mathcal{U}(F)$, if $F \subset \cup\{U: U \in \mathcal{U}(F)\}$ and $U \cap F \neq \Phi$, for each $U \in \mathcal{U}(F)$.

A family of such covers is denoted by $\alpha(F)$.

For $\mathcal{U}=\{U\}$, let $|\mathcal{U}|$ be the union of all elements belonging to the sets $U \in \mathcal{U}$, i. e. $|\mathcal{U}|=\cup\{U: U \in \mathcal{U}\}$.

A cover $\mathcal{U}'$ refines $\mathcal{U}$ if for each $U \in \mathcal{U}$ there is $U' \in \mathcal{U}'$ such that $U' \subset U$ and that $|\mathcal{U}'| \subset |\mathcal{U}|$, and then we write $\mathcal{U} =\!\leqslant \mathcal{U}'$.

These two definitions are well-known and we only want to take them here in the above sense. It is easily seen that a family $\alpha(F)$ of covers of a set F is a partially ordered set with respect to the relation $=\!\leqslant$, i. e.

$$\mathcal{U}_1(F) =\!\leqslant \mathcal{U}_2(F)$$

if $\mathcal{U}_2(F)$ refines $\mathcal{U}_1(F)$.

The class $\{\alpha(F): F \in 2^x\}$ shall be denoted by α (analogy with the symbols $f(x)$ and f for a function and its value at x), and we shall suppose that the following axiom is satisfied:

[1)] Read at the Congress of mathematicians, physicists and astronomers of Yugoslavia in Sarajevo, 3 — 10 Oct. 1965.

(α_1) *For each $F \in 2^x$, $\alpha(F)$ is a directed set with respect to the relation* $=\leqslant$.

It is easy to verify that the axiom (α_1) is satisfied in the following examples:

Example 1. $\alpha(F)$ is the family of all $\mathcal{U}(F)$, where

$$\mathcal{U}(F)=\{U: U \supset F\}.$$

Example 2. $\alpha(F)$ is the family of all $\mathcal{U}(F)$, where $\mathcal{U}(F)$ is a finite cover of F.

Example 3. $\alpha(F)$ is the family of all $\mathcal{U}(F)$, such that $\mathcal{U}(F)$ consists of a finite number of open sets together with the whole X.

Example 4. $\alpha(F)$ is the family of all $\mathcal{U}(F)$, where

$$\mathcal{U}(F)=\{U_0; U_1, \ldots, U_n\}.$$

The set U_0 contains F and the others intersect F.

Example 5. $\alpha(F)$ is the family of all locally finite covers of F

Now, we can consider α as a function with the domain 2^x and the range a subclass of the class of all open coverings of the sets $F \in 2^x$. Given α, we follow Michael [3], or indirectly Vietoris [6], in defining a topology on 2^x. For $F_0 \in 2^x$ and $\mathcal{U}(F_0)$, let

$$\mathcal{U}^*(F_0)=\{F: F \in 2^x,\ F \subset |\mathcal{U}|,\ F \cap U \neq \Phi, \text{ for } U \in \mathcal{U}(F_0)\}.$$

The family of all $\mathcal{U}^*(F_0)$, $\mathcal{U}(F_0) \in \alpha(F_0)$ shall be denoted by $\alpha^*(F_0)$ and the class of all $\alpha^*(F)$, $F \in 2^x$ by α^*.

Proposition 1. *The collection $\alpha^*(F)$ is a base of open neighborhoods of $F \in 2^x$ for a topology on 2^x,*

Proof. 1. $F_0 \in \mathcal{U}^*(F_0)$, for $F_0 \subset |\mathcal{U}|$ and $F_0 \cap U \neq \Phi$ for each $U \in \mathcal{U}(F_0)$. 2. Let $\mathcal{U}_1^*(F_0)$ and $\mathcal{U}_2^*(F_0)$ be two neighborhoods of F_0. Then, according to the axiom (α_1), there is a cover $\mathcal{U}_0(F_0)$ refining both $\mathcal{U}_1(F_0)$ and $\mathcal{U}_2(F_0)$. So, for $\dot{\tau}=1,2$:

$$F \in \mathcal{U}_0^*(F_0) \Rightarrow F \subseteq |\mathcal{U}_0| \text{ and } F \cap U \neq \Phi,\ U \in \mathcal{U}_0$$
$$\Rightarrow F \subseteq |\mathcal{U}_{\dot{\tau}}| \text{ and } F \cap U \neq \Phi,\ U \in \mathcal{U}_1 \cup \mathcal{U}_2,$$

since for each $U \in \mathcal{U}_{\dot{\tau}}$ there is a $U_0 \in \mathcal{U}_0$ such that $U_0 \subset U$. 3. Being $F \subset |\mathcal{U}|$ and $F \cap U \neq \Phi, U \in \mathcal{U}$ for each $F \in \mathcal{U}^*(F_0)$ it follows that $\mathcal{U}^*(F_0)=\mathcal{U}^*(F) \in \alpha^*(F)$.

The topology which, according to the Proposition 1., exists on 2^x, we, shall call α^* — *topology* and the pair $(2^x, \alpha^*)$ α^* — *hyperspace of X*.

The following proposition, which is easily proved, shows how some of the well-known topologies on 2^x can be defined in the same way, choosing only different types of covers $\mathcal{U}(F)$.

Proposition 2. *For a topological space X, α^* — topology on 2^x induced by the systems of covers in the given examples* 1—4 *is identical with*

a) *the topology of the space $\varkappa X$ of V Ponomarev* [4].

b) *the finite topology of Michael* [3],

c) *the topology of the classical space of closed sets* (V. Ponomarev [4]).

d) *the hemi-semi-finite topology of Michael* [3] *respectively.*

Let $\dot{\tau}: X \to 2^x$ be such a mapping that $\dot{\tau}(x)=\{x\}$ (where $\{x\}$ is the singleton x). Then, according to E. Michael, an α^* — topology is *admissible* if i is a homeomorphism. It seems reasonable to add to the axiom (α_1) the following one.

(α_2 *For all* $F \in 2^x$, *each* $\mathcal{U}(F) \in \alpha(F)$ *is a point finite cover.*

Both of these axioms (α_1) and (α_2) shall be supposed in the sequel. Now, we can prove the following

Proposition 3. *The mapping* $\dot{\tau}: X \to 2^x$ *is continuous.*

Proof. For a fixed $\{x\} \in 2^x$ and an arbitrary cover $\mathcal{U}(\{x\})$ in view of (α_2), $\mathcal{U}(\{x\})$ must be a finite cover, i. e.

$$\mathcal{U}(\{x\}) = \{U_1, U_2, \ldots, U_n\}.$$

Taking into account that $\{U_0\}$, $U_0 = \bigcap_{\dot{\tau}=1}^{n} U_{\dot{\tau}}$ as a cover which refines $\mathcal{U}(F)$ then $\{|\mathcal{U}_0|\}^* \cap X$ is a neighborhood of x in X which is mapped into $\mathcal{U}^*(\{x\})$.

From the proof of the preceeding proposition we have the following.

Lemma 1. If $B_x = \{U\}$ *is a base of neighborhoods of the point* $x \in X$, *then* $B^*_{\{x\}} = \{U^*\}$ *is the base of neighborhoods of* $\{x\} \in 2^x$.

If we suppose the following axiom

(α_3) *For an arbitrary open set U containing* $x \in X$, *there is* $\mathcal{U}(\{x)\} \in x(\{x\})$ *such that* $U \in U(\{x\})$;

then we can prove

Proposition 4. *If* (α_3) *is satisfied* ((α_1) and (α_2) are supposed to be), *then the topology* α^* *is admissable,*

Proof. For an arbitrary neighborhood of x, there is $\mathcal{U}(\{x\})$ containing U. Since $\mathcal{U}(\{x\})$ must be finite $\mathcal{U}(\{x\}) = \{U, U_1, \ldots, U_n\}$ the set $U_0^* \cap X$, $U_0 = U \cap \bigcap^{n} U_{\dot{\tau}}$, is mapped back by $\dot{\tau}^{-1}$ into U, so that $\dot{\tau}^{-1}$ is continuous and by Proposition 3., it follows that i is a homeomorphism.

Consequence of Proposition 4. The topologies induced by the covers in examples 1-4 are admissable,

3. Compactness of 2^x. In this part we show that the compactness of X implies (2^x, α^*) is compact provided the two topologies are „close" to each other. Note that α^*-topology need not be admissable. We use the notion of universal net in proving the following theorem (see J. L. Kelley [1] and [2]).

Theorem. *If X is a compact topological space and if for every* $F \in 2^x$, *the cover* $\mathcal{U}(F) \in \alpha(F)$ *is finite, then the space* (2^x, α^*) *is compact,*

Proof. Suppose X is compact. Let

(1) $$\{F_\alpha : \alpha \in A\}$$

be a universal net in 2^x and let

$$S_\alpha = U\{F_\beta : \beta \geqslant \alpha\}, \ \beta \in A.$$

It is obvious that the family $\{S_\alpha : \alpha \in A\}$ has the finite intersection property and so does the family of closed sets $\{\overline{S}_\alpha : \alpha \in A\}$. Since X is compact, the set

$$F_0 = \cap \{\overline{S}_\alpha : \alpha \in A\}$$

is nonempty and closed. We intend to prove that F_0 is a cluster point of the net (1). Let

$$\mathcal{U}(F_0) = \{U_1, \ldots, U_n\}$$

be a cover of F_0 which is supposed to be finite. Then,

$$F_\alpha \subset |\mathcal{U}(F_0)| = \bigcup_{i=1}^{n} U_i , \tag{2}$$

from an index on. In the opposite case there would exist a cofinal subset A_0 of A, such that

$$F_\alpha \cap |\mathcal{U}(F_0)|' \neq \Phi, \ \alpha \in A_0.$$

Choosing then $x_\alpha \in F_\alpha \cap |\mathcal{U}(F_0)|'$, the net $\{x_\alpha ; \ \alpha \in A_0\}$ is in the closed set $|\mathcal{U}(F_0)|'$ which is also compact. So, this net has a cluster point $Z \notin |\mathcal{U}(F_0)|$ and also $Z \notin F_0$. Now, let O be a neighborhood of z. Then for any $\alpha \in A$ there is a $\beta \geqslant \alpha$ such that $x_\beta \in O$. So, $x_\beta \in F_\beta \subset S_\alpha$, what means that $S_\alpha \cap O \neq \Phi$ for each $\alpha \in A$; or else that $z \in \overline{S}_\alpha$ and the conclusion that $z \in \cap \{\overline{S}_\alpha : \alpha \in A\} = F_0$ is a contradiction, what proves (2). Further on, choose $x_1 \in F_0 \cap U_1$. Then, $x \in \overline{S}_\alpha$ and $S_\alpha \cap U_1 \neq \Phi$ for each $\alpha \in A$, what means that there is a cofinal subset A_1 of A such that $F_\alpha \cap U_1 \neq \Phi$ for $\alpha \in A_1$. Let

$$\tilde{U}_1 = \{F : F \in 2^x \text{ and } F \cap U_1 \neq \Phi\}.$$

We have shown that the net (1) is frequently in U_1 and being universal, it must be eventually in $\tilde{U}_1$. This means that there is an index $\alpha_1 \in A$ such that

$$F_\alpha \cap U_1 \neq \Phi, \text{ for } \alpha \geqq \alpha_1.$$

It is proved in the same way the existence of an $\alpha_i \in A$ such that

$$F_\alpha \cap U_i \neq \Phi, \text{ for } \alpha \geqq \alpha_i$$

where $i = 1, 2, \ldots, n$. Choosing $\tilde{\alpha} \in A$ to be greater then all α_i and that for $\alpha > \tilde{\alpha}$ the relation (2) is satisfied; we see that the net (1) is eventually in the arbitrary neighborhood of F_0, what implies its convergence to F_0. Since each universal net in 2^x is convergent this space is compact.

Consequence 1. *The hyperspaces determined by covers given in examples 1-4 are compact.*

In our next example we intend to obtain the topological product space as the subspace of a hyperspace

Example 6. Let X be a topological space and $\mathcal{D} = \{D\}$ a decomposition of this space into closed subsets. Let $\mathcal{U}(\mathcal{D})$ be the family of open sets which are complements of a finite collection of closed sets each belonging to a $D \in \mathcal{D}$. If $\alpha : 2^x \to \mathcal{U}$ i. e. all covers are chosen from $\mathcal{U}$ and if two axioms (α_1) and (α_2) are satisfied then we shall call $(2^x, \alpha^*)$ the *hyperspace* $\mathcal{D}$. There are, of course, more different hyperspaces of this kind, all depending upon the choice of covers $\mathcal{U}(F)$ and the decomposition $\mathcal{D}$.

Let $\{X_\zeta; \ \zeta \in Z\}$ be a collection of compact Hausdorff spaces and

$$X = \Sigma \{X_\zeta; \ \zeta \in Z\}$$

their topological sum. Each X_ζ is closed (and open) in X. Let $\tilde{X} = X \cup (\infty)$ be one point compactification of X. Now, we are going to give a more specific example of a $\mathcal{D}$ hyperspace and we point out that the main difference between this example and those given in the part 2., is in the fact we use here not the whole class of open sets but its subclasses.

Let $X^{\zeta_1 \dots \zeta_n} = \tilde{X} \setminus \left(\overset{n}{\underset{\tau=1}{U}} X_{\zeta_\tau} \right)$ and $O_\zeta \subset X_\zeta$ be open. Then,

$$\alpha (F) = \{ X^{\zeta_1 \dots \xi_n};\ O'_{\zeta_1}, \dots, O'_{\xi_m} \}.$$

If $\mathcal{D} = \{D\}$ is decomposition of X consisting of X_ζ and ∞ let $\mathcal{G}$ be the family of $F \in 2^{\tilde{x}}$ such that $F \cap D$ is single point. Then, we have

Lemma 2. *The space* $(\mathcal{G}, \alpha^*)$ *is homeomorphic to the topological product space* X $\{X_\zeta, \zeta \in Z\}$ *and* $\mathcal{G}$ *is a closed subset of* $2^{\tilde{x}}$,

Proof. The proof of the first part is entirely straightforward and so omitted. We shall prove the second part. If $F_0 \in 2^{\tilde{}}_x \setminus \mathcal{P}$ and is such that $F_0 \cap X_\zeta = \Phi$ for some $\zeta \in Z$, then the neighborhood $\mathcal{U}^* (F_0)$, $\mathcal{U} (F_0) = \{X^\zeta\}$ does not contain any $F \in \mathcal{G}$. Otherwise, $F_0 \cap X_\zeta$ contains more then one point for some ζ, say, $x, y \in F_0 \cap X_\zeta$. Taking two disjoint neighborhoods O'_ζ and O''_ζ of x and y respectively, we find that $\mathcal{U}^* (F_0)$, $\mathcal{U} (F_0) = \{X^\zeta;\ O'_\zeta, O''_\zeta\}$ is disjoint from $\mathcal{U}^* (F_0)$).

Being $2^{\tilde{x}}$ compact as it follows from the Theorem and using the above lemma we have

Consequence 2. (Tychonoff) *The topological product of compact spaces is compact,*

Let, now, X be a normed linear space, 2^x its hyperspace with the topoogy of Michael; then we can prove this

Lemma 3. If $\{C_\alpha;\ \alpha \in A\}$ *is a net of convex sets converging to* $F_0 \in 2^x$, *then* F_0 *is convex.*

Proof. Suppose F_0 is not convex. Then, there are two points $x, y, \in F_0$ and a $Z \in \dot{\tau}\, n t v\, x y$ such that $Z \notin F_0$. Choose the sphere $\overline{S} (Z, \varepsilon) \subset F'_0$ and we can suppose without loss of generality $Z = 0$. Taking

$$\mathcal{U} (F_0) = \{ \overline{S} (Z, \varepsilon)',\ x + S (Z, \varepsilon),\ y + S (Z, \varepsilon) \}$$

and choosing a less ε, if necessary, so to be $x + S (Z, \varepsilon)$ and $y + S (Z, \varepsilon)$ contained in $\overline{S} (Z, \varepsilon)'$. If $C_\alpha \in \mathcal{U}^* (F_0)$, there will exist $a \in C_\alpha \cap (x + S (Z, \varepsilon))$, $b \in C_\alpha \cap (y + S (Z, \varepsilon))$. Then, $Z = \alpha x + \beta y$, $\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$, so we have

$$\begin{aligned} \| \alpha a + \beta b \| &= \| \alpha (a - x) + \beta (b - y) \| \\ &\leqslant \alpha \| a - x \| + \beta \| b - y \| < \varepsilon \end{aligned}$$

Since C_α is convex, $\alpha a + \beta b \in C_\alpha$ and $\alpha a + \beta b \notin \overline{S} (Z, \varepsilon)'$, what contradicts $C_\alpha \to F_0$.

Our next consequence is known as Blaschke Convergence Theorem (see F. Valentine [5]).

Consequence 3. *Let* $\mathcal{M}$ *be a uniformly bounded infinite collection of closed convex sets in a Minkowski space* L_s. *Then* $\mathcal{M}$ *contains a sequence which converges to a nonempty compact convex set.*

Proof. Let S be a closed sphere containing $\mathcal{M}$. Then, S is compact, so that 2^s is also compact with respect to the topology of Michael (which coincides with topology induced by Hausdorff metric). According to the Proposition 4.5 of Michael [3], 2^x is first countable so that there is a sequence in $\mathcal{M}$ converging to a $C_0 \in 2^S$. C_0 is convex as it follows from Lemma 3.

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